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BOOK OF ABSTRACTS

Contents

Plenary Speakers

Classes of “big” spaces from characterizations of metrizability

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Winner of the “Franco Montagna" prize.

Metrizable spaces are a key concept in numerous areas of mathematics, including descriptive set theory. Consequently, many different characterizations of metrizability have been proposed.

Each characterization brings light on the crucial topological properties that guarantee the desirable behavior of metric spaces. These properties are particularly valuable in generalized descriptive set theory, where they help define new classes of non-first-countable (and thus non-metrizable) topological spaces. This enables the extension of classical descriptive set theory results pertaining to Polish spaces to these new classes of spaces.

Various theorems offer different properties that define distinct classes of spaces, leading to questions about the existence of a preferable class and the sufficiency of specific properties for deriving certain theorems.

In this talk, I will examine different characterizations of metrizability, compare them, and show how some results from classical descriptive set theory can be recovered from certain properties and not from others.

This is joint work⋆ with Luca Motto Ros.

Inquisitive neighborhood logic

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In this talk, I will first introduce the motivation and the key ideas of the ongoing research program on inquisitive modal logic. I will then present a particular system of inquisitive modal logic designed to talk about properties of neighborhood structures, whose basic modality is a sort of strict conditional quantifying over neighborhoods. I will discuss two concrete interpretations of this logic, characterize its expressive power in terms of a natural notion of bisimulation, and present complete axiomatizations for some salient classes of frames.

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Differential linear logic: from semantics to syntax

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In 1969, Dana Scott met Christopher Strachey in Oxford. Together they created a new branch of fundamental computer science: *denotational semantics*. Since the mid 60's, Strachey was advocating a *mathematical semantics* of programs independent from their implementation on actual computers. Due to the increasing number of computer architectures and of programming languages, the need of such a semantics became indeed more and more critical. Denotational semantics is based on new ideas of Scott which allowed him in 1968 to build the complete lattice D_∞, solving a long-standing problem: find a mathematical interpretation of pure lambda-terms allowing to see them as the functions they are intuitively denoting.

Denotational semantics was mainly considered of a topological nature until 1986, when Jean-Yives Girard discovered *linear logic*, recasting proofs and programs in a setting much closer to linear algebra (often in infinite dimension, where topology may become necessary). The denotational models of linear logic are intuitively categories of linear morphisms equipped with an exponential, a modality (technically, a comonad) allowing them to host also non-linear morphisms which are intuitively to be considered as analytic functions. One major feature of linear logic is to deeply relate algebraic linearity with operational linearity: a program is linear when it uses completely and exactly once its argument.

Linear logic has a pivotal rule called *dereliction* allowing to see a linear morphism as an analytic one, simply forgetting linearity. In most models of linear logic, this rule has a kind of inverse which can be understood as an operation of "differentiation at 0". In the early 2000's I observed that, in severeal interesting models of linear logic, this *codereliction* rule is complemented with additional rules, symmetric to the usual structural rules associated in linear logic with the exponential. Altogether, these new rules allow to compute the differential of an arbitrary proof of linear logic. Starting from this observation, with Laurent Regnier, we introduced *differential linear logic* and the *differential lambda-calculus*.

In this talk, I will present this global conceptual framework and explain how we developped the differential lambda-calculus, introducing in particular the Taylor expansion of lambda-terms which uses iterated differentiation of terms and provides a fine-grain algebraic theory of program approximations. I will also discuss the need to equip the differential lambda-calculus with an addition operation on terms, the operational and denotational consequences of this operation, and how it can be controlled in the recently discovered setting of *coherent differentiation*.

The strength of a theorem of Rival and Sands

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In 1979 Ivan Rival and Bill Sands proved that each infinite graph G has an infinite subgraph H such that each vertex of G is adjacent to none or to one or to infinitely many vertices of H. This statement, showing the existence of a substructure with some property in every infinite graphs, resembles Ramsey's theorem for pairs, which guarantees the existence of a complete or a totally disconnected subgraph in each infinite undirected graph.

We investigated the strength of this statement (restricted to countable graphs) from the viewpoint of Weihrauch reducibility and reverse mathematics. During the talk variants of this theorem and their strength are also presented.

This is joint work with Paul Shafer and Giovanni Soldà.

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Model theory of complex analytic functions: quasiminimality and existential closedness

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A (model-theoretic) structure S is *quasiminimal* if all the definable subsets of S are countable or have countable complement. In the 1990s, Zilber made a striking conjecture [3] predicting that the complex numbers seen as a structure in the language of rings expanded by a symbol for the complex exponential function should be quasiminimal.

This conjecture soon turned out to be related to several results and open problems from transcendental number theory and Diophantine geometry, and in turn opened up new lines of research, both in logic and geometry. In logic, it led to the development of a theory for quasiminimal structures, with categoricity theorems for classes of quasiminimal structures axiomatizable in some infinitary logic. In geometry, *existential closedness* problems were introduced: these are concerned with the existence of solutions to systems of equations involving polynomials and certain analytic functions, such as the complex exponential or the modular j invariant. Bays and Kirby's result from [1] was an important milestone, establishing that the quasiminimality conjecture for $\exp$ (a purely model-theoretic statement) would follow from the *exponential-algebraic closedness conjecture*, which predicts the existence of solutions of appropriate systems of polynomials and exponentials.

In this talk I will survey some of these conjectures and the relations between them, and some partial results towards them. In particular, I will discuss the result of Kirby and myself [2] about quasiminimality of the complex field expanded with multivalued *power functions*, where for each $\lambda \in \mathbb{C}$ the power function $w \mapsto w^\lambda$ on $\mathbb{C}^\times$ maps $w \in \mathbb{C}^\times$ to the set of determinations of $\exp(\lambda \log w)$.

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Stone duality for spectral sheaves and the patch monad

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Sheaf representations may be viewed as generalising Stone's representation theorem and the ensuing duality for Boolean algebras. In this setting, we establish a duality between global sheaves on spectral spaces and a category of very simple idempotent semigroups known as right distributive bands. This is a sheaf-theoretical extension of classical Stone duality between spectral spaces and bounded distributive lattices. The topology of a spectral space admits a refinement, the so-called patch topology, giving rise to a monad on sheaves over a fixed spectral space which we call the patch monad. Under the duality just mentioned the algebras of this patch monad are shown to correspond to distributive skew lattices, where skew lattices are a non-commutative variant of lattices originating in quantum logic and operator algebra. The research reported on in this talk is joint work with Clemens Berger.

Borel-definable algebraic topology

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Recently, Bergfalk, Panagiotopoulos, and I have introduced refinements of classical algebraic invariants endowed with additional information of descriptive set-theoretic nature. I will present an overview of applications of such Borel-definable invariants to algebra and topology, obtained jointly with Bergfalk, Casarosa, Codenotti, Meadows, Sarti, and Panagiotopoulos.

On point to set principles, normality, and algorithmic randomness

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Effective and resource-bounded dimensions were defined by Lutz in [5] and [4] and have proven to be useful and meaningful for quantitative analysis in the contexts of algorithmic randomness, computational complexity and fractal geometry (see the surveys [2, 6, 3, 10] and all the references in them).

The point-to-set principle of J. Lutz and N. Lutz [7] fully characterizes Hausdorff and packing dimensions in terms of effective dimensions in the Euclidean space, enabling effective dimensions to be used to answer open questions about fractal geometry, with already an interesting list of geometric measure theory results (see [9, 8]).

Finite state dimension [1] is the lowest level effectivization of Hausdorff dimension and is closely related to Borel normality. In this talk I will review its main properties, prove a new characterization in terms of information content approximated at a certain precision, and consider new generalizations of normality. I will then prove a finite-state dimension point to set principle [11].

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Infinite Games and Large Cardinals: Attacking Independence by Connecting the Hierarchies

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The independence phenomenon is a central theme in set theory. Moreover, starting from the Continuum Problem, there are numerous statements in various areas of mathematics that can neither be proven nor disproven in the usual axiomatic framework given by the Zermelo Fraenkel Axioms. The most promising approach to attack this issue is by studying extensions of the usual axiomatic framework, their connections, and their impact on previously independent statements. Two famous examples in set theory are the hierarchies given by the determinacy of infinite two-player games and large cardinals. The deep connection between these two hierarchies forms the backbone of inner model theory. We outline this connection and discuss recent progress as well as important open questions in the area.

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Rigidity, Rigidity, Rigidity!

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We are interested in the effects of additional set theoretic axioms on quotient structures and their isomorphisms. Our focus is on rigidity, measured in terms of existence (or rather non-existence) of suitably non-trivial isomorphisms of the quotients in question. Consider for example the Boolean algebra $\mathcal{P}(\omega)/\mathrm{Fin}$: Forcing axioms imply that all of its automorphisms are trivial, while under the Continuum Hypothesis this rigidity fails. This behavior is the template around which this area of work revolves, and in this talk we consider some of its generalizations. We present a variety of situations where analogous patterns persist, such as (reduced products of) Boolean algebras, graphs, groups, or linear orders, but also more analytic objects such as Čech–Stone remainders, C^*-algebras or even objects constructed from coarse geometrical data.

Contributed talks

Vietoris endofunctor for closed relations and its de Vries dual

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SPEAKER: Marco Abbadini.

We generalize the Vietoris endofunctor to the category of compact Hausdorff spaces and closed relations and describe the dual endofunctor on the category of de Vries algebras and subordinations. This presentation is based on [3].

Taking the Vietoris hyperspace $\mathbb{V}(X)$ of a compact Hausdorff space X defines an endofunctor $\mathbb{V}$ on the category KHaus of compact Hausdorff spaces and continuous functions. On morphisms, a continuous function $f\colon X \to Y$ is mapped to the function $\mathbb{V}(f)\colon \mathbb{V}(X) \to \mathbb{V}(Y)$ which maps a closed subset F of X to the image $f[F]$ of F under f.

The larger category $\mathsf{KHaus}^{\mathsf{R}}$ of compact Hausdorff spaces and closed relations has been investigated in various works [11, 8, 10, 5, 1]. One appealing feature of $\mathsf{KHaus}^{\mathsf{R}}$ is that it is self-dual. We generalize the Vietoris endofunctor to an endofunctor $\mathbb{V}^{\mathsf{R}}\colon \mathsf{KHaus}^{\mathsf{R}} \to \mathsf{KHaus}^{\mathsf{R}}$. For a subordination $R \subseteq X \times Y$, we define $\mathbb{V}^{\mathsf{R}}(R)$ by generalizing the well-known Egli-Milner order: for all closed subsets $F \subseteq X$ and $G \subseteq Y$,

$$F \; \mathbb{V}^{\mathsf{R}}(R) \; G \iff G \subseteq R[F] \text{ and } F \subseteq R^{-1}[G],$$

where $R[F]$ is the R-image of F in Y and $R^{-1}[G]$ is the R-preimage of G in X. We show that this defines an endofunctor $\mathbb{V}^{\mathsf{R}}\colon \mathsf{KHaus}^{\mathsf{R}} \to \mathsf{KHaus}^{\mathsf{R}}$ which restricts to the Vietoris endofunctor $\mathbb{V}\colon \mathsf{KHaus} \to \mathsf{KHaus}$ and commutes with the self-duality of $\mathsf{KHaus}^{\mathsf{R}}$.

De Vries duality [7] is a duality for KHaus which associates with each compact Hausdorff space X the boolean algebra $\mathcal{RO}(X)$ of regular opens of X equipped with the proximity relation given by $U \prec V$ iff $\mathsf{cl}(U) \subseteq V$. This yields a duality between KHaus and the category DeV of *de Vries algebras*, i.e. pairs $(B, \prec)$ where B is a complete boolean algebra and $\prec$ is a proximity relation on B. A direct pointfree construction of the endofunctor $\mathsf{DeV} \to \mathsf{DeV}$ dual to $\mathbb{V}\colon \mathsf{KHaus} \to \mathsf{KHaus}$ remained an open problem [4, p. 375]. We resolve this problem as follows.

In [1] we extended de Vries duality to $\mathsf{KHaus}^{\mathsf{R}}$. Let $\mathsf{Stone}^{\mathsf{R}}$ be the full subcategory of $\mathsf{KHaus}^{\mathsf{R}}$ consisting of Stone spaces. Stone duality extends to an equivalence between $\mathsf{Stone}^{\mathsf{R}}$ and the category BA^{S} with boolean algebras as objects and subordination relations as morphisms [6, 9, 1]. This yields an equivalence between $\mathsf{KHaus}^{\mathsf{R}}$ and a category whose objects are pairs (B, S) where B is a boolean algebra and S is a subordination relation on B satisfying axioms generalizing the axioms of an S5-modality. Because of this connection, we termed the pairs (B, S) S5-*subordination algebras* and denoted the resulting category by $\mathsf{SubS5^S}$ [1]. The inclusion $\mathsf{DeV^S} \hookrightarrow \mathsf{SubS5^S}$ of the full subcategory $\mathsf{DeV^S}$ consisting of de Vries algebras is an equivalence, with quasi-inverse obtained by generalizing the MacNeille completion to S5-subordination algebras [2].

In [12], the endofunctor $\mathbb{K}$ on boolean algebras dual to the Vietoris endofunctor $\mathbb{V}$ on Stone spaces was defined. We lift $\mathbb{K}$ to an endofunctor $\mathbb{K}^\mathsf{S}$ on $\mathsf{BA^S}$ equivalent to $\mathbb{V}^\mathsf{R}$ on $\mathsf{Stone^R}$. Finally, we lift $\mathbb{K}^\mathsf{S}$ to an endofunctor on $\mathsf{SubS5^S}$ equivalent to $\mathbb{V}^\mathsf{R}$ on $\mathsf{KHaus^R}$. Composing it with the MacNeille completion yields an endofunctor on $\mathsf{DeV^S}$ equivalent to $\mathbb{V}^\mathsf{R}$. This solves the problem mentioned above in the category $\mathsf{SubS5^S}$, in its full subcategory $\mathsf{DeV^S}$, and finally in DeV via a duality between DeV and a wide subcategory of $\mathsf{DeV^S}$.

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On Propositional Dynamic Logic and Concurrency

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SPEAKER: Matteo Acclavio.

Logic, and in particular proof theory, offers several approaches to reason about different computational properties of programs. The Curry-Howard correspondence represents a program by a proof, thus providing a strong foundation for the development of type systems [29, 3, 8]. In logic programming, a program is an inference system, which allows for using proof search as the means of execution [17, 19]. In *dynamic logic* (DL), programs are part of the language of formulas itself, which enables the direct use of the logic to reason about the semantics of programs [10]. Under the latter view, the purpose of programs is to change the truth value of a formula. At the syntactical level, each program a defines the modalities $[a]$ and $\langle a \rangle$ and a formula $[a]\,\phi$ is interpreted as "every state reached after executing a satisfies the formula ϕ" while a formula $\langle a \rangle\,\phi$ is interpreted as "there is a state reached after executing a satisfying the formula F". This idea has been of profound inspiration in the field of formal verification [28, 6]. In this work, we are interested in the propositional fragment of dynamic logic (Propositional Dynamic Logic, or PDL) [16].

PDL and the concurrency problem

While PDL has been successfully applied to the study of sequential programs, extending this approach to concurrent programs has been proved to be challenging. In standard PDL, a program is represented by a regular expression that describes its set of possible traces. In other words, programs are elements of a free Kleene algebra. This representation of programs is satisfactory when reasoning about sequential programs, because one obtains that the theory of equational reasoning for Kleene algebras is a complete system for reasoning about *trace equivalence* [12, 15, 13, 27]. Trace equivalence is therefore captured by logical equivalence in PDL:

$$a \text{ and } b \text{ have the same traces} \quad \text{iff} \quad \vdash_{\mathsf{PDL}} [a]\,\phi \Leftrightarrow [b]\,\phi\,. \qquad (1)$$

However, the case of concurrent programs with an interleaving semantics is more problematic. In the presence of interleaving, one expects traces differing by interleaving to be equivalent modulo equations of the form $a; b = b; a$ (called *commutations*). Unfortunately, the word problem in a Kleene algebra enriched with an equational theory containing such commutations is known to be undecidable [14], which makes undecidable checking whether two modalities in PDL are the same. This is proven by reducing the Post correspondence problem to the word problem by combining sequential composition, iteration, and commutations.

As a consequence of this problem, applications of PDL to concurrency fall short of the expected level of expressivity from established theories, like CCS [20] and the π-calculus [21]. For example, previous works lack nested parallel composition, synchronisation, or recursion [18, 4, 25, 26, 24, 3]. In general, adding any new concurrency feature (e.g., a construct in the language of programs or a law defining its semantics) requires great care and effort in establishing the meta-theoretical properties of the logic. The result: a literature of various PDL, all independently useful, but with different limitations and dedicated technical developments.

In this talk

We discuss the result in [2], where we develop *operational propositional dynamic logic* (OPDL). The key innovation of OPDL is to distinguish and separate reasoning on programs from reasoning on their traces. Thanks to this distinction, we circumvent previous limitations and finally obtain a PDL that can be applied to established concurrency models, such as CCS [20] and choreographic programming [22]. Crucially, OPDL is a general framework: it is parameterised on the operational semantics used to generate traces from programs, yielding a simple yet reusable approach to characterise trace reasoning.

After recalling the axiomatization and semantics of PDL, we provide a proof of its soundness and completeness with respect to the non-wellfounded sequent calculus introduced in [7]. For this purpose, we provide the first cut-elimination result for this non-wellfounded calculus, by adapting the technique developed in [1].* This allows us to prove our results by reasoning on the axiomatisation and the sequent system, without directly relying on semantic arguments.

Then, we extend PDL with an additional axiom allowing us to encapsulate an operational semantics for a set of programs into the trace reasoning. We call the resulting logic *operational propositional dynamic logic* (or OPDL), providing a general framework encompassing various previous works [18, 4, 9].

We then conclude by instantiating OPDL for for Milner's CCS [20] and Montesi's latest presentation of choreographic programming [23].

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*A cut-elimination result for another sequent calculus for PDL is provided in [11], but that calculus is fundamentally different: it employs nested sequents and contains rules with an infinite number of premises.

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On Proof Equivalence and Combinatorial Proofs

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SPEAKER: Matteo Acclavio

Proof theory is the branch of logic studying proofs as mathematical objects. Since its appearance at the end of 19th century, this mathematical discipline has seen enormous progress, also thanks to its strong connection with many areas of theoretical computer science, providing the foundation for, among others, automated reasoning, formal verification and the theory of programming languages. Proofs expressed in various formalisms, as well as transformations of proofs (e.g., composition and normalization), have been used to provide different computational paradigms to interpret proofs as abstractions of computer programs.

In proof theory we find an abundance of formalisms to define the same mathematical objects, as can be observed in many mathematical fields. However, what makes proof theory an anomaly is the struggle to provide a clear definition of the equivalence between its basic objects, not only when expressed in two distinct formalisms, but also when two proofs are represented in the same proof system. The fundamental question "*when two proofs are the same?* " has not been asked formally in recent times. Besides the intuition that certain proofs may not be the same, find a criterion for simplicity of proofs (also known as the 24th Hilbert's problem*) or even deciding when two proofs are the same poses various challenges, not only when comparing proofs in different formalisms, but also proofs in a same formalisms. In fact, one can say that proof theory, in its current form, is not the theory of proofs but the theory of proof systems and their properties.

The two main approaches to provide a notion of canonical representative for proofs relay on the notions of *normalization* and *generality*. The first approach aims at reducing proofs to *normal forms* (e.g., by cut-elimination in sequent calculus and by removing elimination-rules in natural deduction), while the second aims at providing a method to identify proofs by removing non-relevant information. While the first approach has been successfully applied to establish results such as the consistency of the Arithmetic, as well as to provide a solid theoretical background to functional programming via the Curry-Howard-Lambek correspondence, it has some limits due to the possible loss of information during normalization (see, e.g., [3]). The intuition behind generality is that the order of certain inferences in a proof is irrelevant as soon as they could be performed "concurrently", but that the syntaxes we use to represent proofs, similarly to the natural language, force us to choose a specific order. In this regard, focused proof systems [2] considerably reduce the rule permutations in the sequent calculi by grouping rules into phases and could be considered a good candidate for a notion of proof equivalence. Nevertheless, because of the sequential nature of the focused systems, derivations differing for the order of phases are still considered different proofs even if they could be transformed the

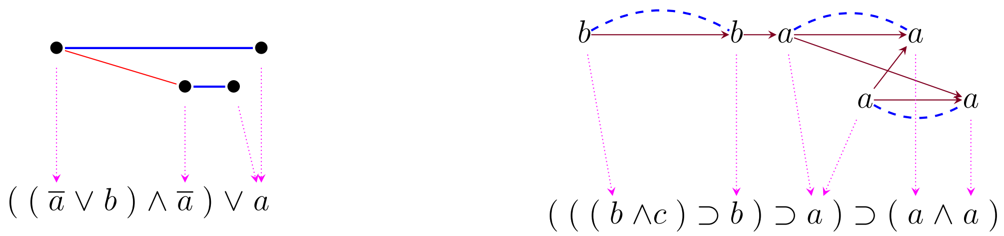
*After the relatively recent discovery of Hilbert's notes by Thiele [10].

one into the other via rule permutations. For this reason, syntaxes such as Girard's *proof nets*[7], Guglielmi's *open deductin* [6], and Hughes *combinatorial proofs* [7, 8] seems the best option for the study of proof equivalence.

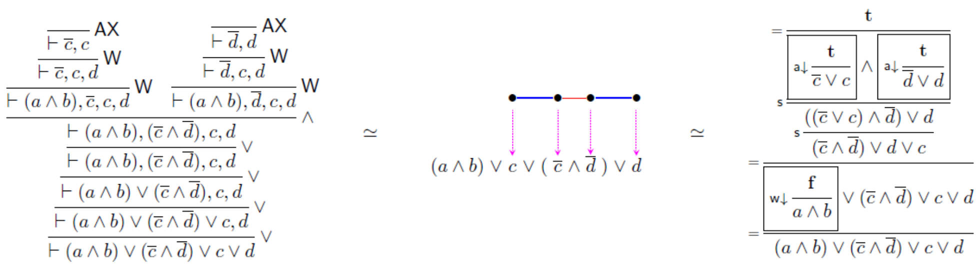
This talk focuses on the syntax of combinatorial proofs, which we use to provide the following notion of proof equivalence.

Two proofs are the same iff they have the same combinatorial proof. (2)

Combinatorial proofs are a graphical syntax where proofs are encoded by specific maps (encoding structural rules) from graphs provided with an additional edge relation (encoding linear proofs) to graphs encoding formulas. Moreover, combinatorial proofs are a proof system in the sense of [4], since it is possible to check in polynomial time whether these objects satisfy the topological conditions ensuring that they are encodings of correct proofs.



Combinatorial proofs capture rules permutations (including those permutations which are required during the cut-elimination procedure in sequent calculus) and allow to compare proofs in different proof formalisms (see Figure 2), such as sequent calculus [8, 7], calculus of structures [9], resolution calculus and analytic tableaux [1].



After providing an introduction to the syntax of combinatorial proofs, we discuss the key features of this formalism, and we analyze the notion of proof equivalenced by Equation (2) in various proof systems. We then provide an overview of the state of the art of combinatorial proofs, providing intuitions on how the combinatorial syntax can be extended and refined to express proofs in relevant, modal, intuitionistic and first-order logic.

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Universal completeness theorems in algebra and logic

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SPEAKER: Paolo Aglianò.

Let ρ be any algebraic language and let $\mathbf{T}_\rho(\omega)$ be the set of ρ-terms (i.e. the *algebra of formulas*). By a logic $\mathcal{L}$ we mean any substitution invariant finitary consequence relation $\vdash\subseteq \mathcal{P}(\mathbf{T}_\rho(\omega)) \times \mathbf{T}_\rho(\omega)$. If Σ, Δ are finite sets of formulas a **clause** (a.k.a. a *multiple conclusion rule*) is a expression of the form $\Sigma \Rightarrow \Delta$; a **rule** is a clause in which $\Delta = \{\delta\}$ is a singleton.

A clause $\Sigma \Rightarrow \Delta$ is **derivable** in $\mathcal{L}$ if $\Sigma \vdash \Delta$; a clause is **admissible** in $\mathcal{L}$ if for every substitution σ:

$$\vdash \sigma(\alpha) \text{ for all } \alpha \in \Sigma \;\text{ implies }\; \vdash \sigma(\beta) \text{ for some } \beta \in \Delta.$$

Not every admissible clause is derivable; a popular example is Harrop's rule for intuitionistic logic

$$\{\neg p \to (q \vee r)\} \;\Rightarrow\; \{(\neg p \to q) \vee (\neg p \to r)\}$$

which is admissible but not derivable. Therefore the problem of determining which sets of admissible clauses (rules) are derivable is paramount. In general if $\mathcal{S}$ is a set of admissible (clauses) rules of a logic $\mathcal{L}$ we may say that $\mathcal{L}$ is **$\mathcal{S}$-complete** if every clause (rule) in $\mathcal{S}$ is derivable. $\mathcal{S}$ of course can be any set, but usually we consider sets that have some logical significance. In particular a clause (rule) $\Sigma \Rightarrow \Delta$ is **passive** if for any substitution σ, $\sigma(\Delta)$ is not a theorem of $\mathcal{L}$; and it is **active** if it is not passive. A logic $\mathcal{L}$ is

- *universally (structurally) complete* if every admissible clause (rule) is derivable;
- *actively universally (structurally) complete* if every active admissible clause (rule) is derivable;
- *passively universally (structurally) complete* if every passive admissible clause (rule) is derivable.

Our techniques we will be mainly algebraic, making use of the so-called Blok-Pigozzi Galois connection. In a few words if $\mathcal{L}$ is an algebraizable logic with equivalent quasivariety semantics Q, then the Blok-Pigozzi connections transforms (sets of) formulas of L into (sets of) equations in Q in a uniform way. In this case clauses of $\mathcal{L}$ correspond to *universal sentences* in Q and rules of $\mathcal{L}$ correspond to quasiequations in Q. As a matter of fact let's say that a universal sentence Φ of Q is derivable in Q if $\mathsf{Q} \vDash \Phi$ and it is admissible in Q if $\mathbf{F}_{\mathsf{Q}}(\omega) \vDash \Phi$ ($\vDash$ is the usual semantic consequence relation). Then if Q is the equivalent quasivariety semantics of $\mathcal{L}$ the Blok-Pigozzi connection sends admissible (derivable) clauses of $\mathcal{L}$ into admissible (derivable) universal sentences of Q. This has two obvious advantages: if $\mathcal{L}$ is algebraizable than we can treat any completeness problem in a purely algebraic fashion; moreover the completeness properties of a quasivariety make sense independently of their logical origin. Moreover for each of these concept there is a *hereditary* counterpart; for instance a quasivariety Q (a logic $\mathcal{L}$) is hereditarily universally complete if all the subquasivarieties of Q (all the finitary extensions of $\mathcal{L}$) are universally complete.

We will discuss mainly universal and active universal completeness and their hereditary counterparts, linking these topics with other interesting concepts such as (weak) projectivity, exactness and, more generally, Ghilardi's *algebraic unification theory* [3]. Here is a list of some of the results we will present in the talk:

Theorem 1. *(see also [2, Proposition 6]) For any quasivariety* Q *the following are equivalent:*

1. Q *is universally complete;*
2. *for every universal class* $\mathsf{U} \subseteq \mathsf{Q}$, $\mathbf{H}(\mathsf{U}) = \mathbf{H}(\mathsf{Q})$ *implies* $\mathsf{U} = \mathsf{Q}$.
3. $\mathsf{Q} = \mathbf{ISP}_u(\mathbf{F}_{\mathsf{Q}}(\omega))$*;*
4. *every finitely presented algebra in* Q *is in* $\mathbf{ISP}_u(\mathbf{F}_{\mathsf{Q}}(\omega))$.

Theorem 2. *If every finitely presented algebra in* Q *is exact then* Q *is universally complete.*

Theorem 3. *If* Q *is universally complete, then* Q *is unifiable.*

Theorem 4 ([2]). *Let* Q *be a locally finite variety of finite type. Then* Q *is universally complete if and only if* Q *is unifiable and has exact unifiers.*

Theorem 5. *Let* Q *be a quasivariety with projective unifiers and such that* $\mathbf{F}_{\mathsf{Q}}$ *is trivial; then* Q *is hereditarily universally complete.*

Theorem 6. *Let* Q *be a quasivariety. The following are equivalent:*

1. Q *is actively universally complete;*
2. *every unifiable algebra in* Q *is in* $\mathbf{ISP}_u(\mathbf{F}_{\mathsf{Q}}(\omega))$*;*
3. *every finitely presented and unifiable algebra in* Q *is in* $\mathbf{ISP}_u(\mathbf{F}_{\mathsf{Q}}(\omega))$*;*
4. *every clause admissible in* Q *is satisfied by all finitely presented unifiable algebras in* Q*;*
5. *for every* $\mathbf{A} \in \mathsf{Q}$*,* $\mathbf{A} \times \mathbf{F}_{\mathsf{Q}} \in \mathbf{ISP}_u(\mathbf{F}_{\mathsf{Q}}(\omega))$.

Theorem 7. *Suppose that* Q *is a quasivariety such that* $\mathbf{F}_{\mathsf{Q}} = \mathbf{F}_{\mathsf{Q}'}$ *for all nontrivial* $\mathsf{Q}' \subseteq \mathsf{Q}$*. If* Q *has projective unifiers then it is hereditarily active universally complete.*

We will also discuss several examples and show that the class of hereditarily active universally complete quasivarieties is strictly smaller then the class of active universally complete quasivarieties. Finally we remark that all the results we have mentioned are contained in [1].

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(Prop)	Instances of propositional tautologies	(MP)	From $\varphi \to \psi$ and φ infer ψ
(M)	$\Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi)$	(C)	$(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$
(K)	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	(N)	$\Box\top$
(Rm)	From $\varphi \to \psi$ infer $\Box\varphi \to \Box\psi$	(Nec)	From φ infer $\Box\varphi$
(Re)	From $\varphi \leftrightarrow \psi$ infer $\Box\varphi \leftrightarrow \Box\psi$		
(I)	From $\varphi \to \psi$ and $\psi \to \chi$ infer $(\Box\varphi \wedge \Box\chi) \to \Box\psi$	(S)	$\Box\top \leftrightarrow \Box\bot$

Table 1: Axioms and rules.

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Non-Monotonic Modal Logics: The Interpolation Rule

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SPEAKER: Thomas Ågotnes.

This abstract is partly based on [5].

1. Weak Modal Logics

Consider the standard modal language: $\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi$, where $p \in \texttt{Prop}$ (a set of atomic propositions). We assume the standard derived propositional connectives, and we write $\Diamond\varphi$ for $\neg\Box\neg\varphi$. The most commonly used axioms and rules for modal logics [3] are shown in the upper part of Table 1. We follow the standard convention of naming axiomatic systems. All the systems we consider are implicity assumed to the axiom (Prop) and the rule (MP). System **E** extends this basic system with the (Re) rule, **EK** extends **E** with axiom (K), and so on. We sometimes abuse notation and also use **L** for the smallest set of formulas that contains the axioms of **L** and is closed under the rules of **L**; we say that a formula is *derivable* if it is contained in **L** and that a rule is *admissible* if **L** is closed under the rule.

We are interested in modal logics that are *not normal* (does not contain both (K) and (Nec) but still are *classical* (admits (Re)), and in particular classical logics that are not *monotonic* (does not contain (M)). All normal modal logics are monotonic. A classical logic is montonic iff (Rm) is admissible.

2. The Interpolation Rule

In this paper we would like to focus on a new, or at least neglected, rule that we call *the interpolation rule* (I) – see Table 1. ψ here acts as an "interpolant" between φ and $\chi^{\star}$.

*Note that unlike in Craig's interpolation theorem [1], the most well-known use of the term "interpolation" in formal logic, there is no assumption about common vocabulary among the involved formulas.

The Interpolation rule is at the same time a strengthening of the Equivalence rule and a weaking of the Monotonicity rule:

Proposition 1. (Re) *is admissible in any logic containing* (I). (I) *is adm. in any logic containing* (Rm).

Thus we have, for example, that $\mathbf{ECK} \subseteq \mathbf{ICK} \subseteq \mathbf{RmCK}$. Both inclusions are strict:

Proposition 2. (I) *is not admissible in* ***EK*** *or* ***ECK****.* (Rm) *is not admissible in* **IC**.

One difference between Equivalence and Monotonicity is that in the presence of the (C) axiom the latter can derive the (K) axiom while the former cannot. As Interpolation is "in between", a natural question is on which side it falls on. The following answers that question.

Proposition 3. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ *is derivable in* ***IC****.*

We can thus position the interpolation rule in the landscape of non-normal modal logics as follows: $\mathbf{EK} \subset \mathbf{ECK} \subset \mathbf{ICK} = \mathbf{IC} \subset \mathbf{RmCK} = \mathbf{EMCK} = \mathbf{EMC}$

As pointed out by [4], modal logics with "the normality schema" (K) but which nevertheless are sub-normal have been "widely neglected" when it comes to logical investigations. One possible reason for this is a perceived lack of applications. We now demonstrate that natural applications exist.

An Application: Secretly Knowing

Let us briefly sketch a natural application of the interpolation rule, in the logic of knowledge and belief [2] – more specifically in the logic of "*secretly* knowing*". A two-agent* *(Kripke) model* $M = (W, R_a, R_b, V)$ consists of a set of states W, a binary relation R_a on W called a's accessibility relation, and the same for b, and a valuation function $V : W \to 2^{\mathsf{Prop}}$.

We will now interpret $\Box\varphi$ as "agent a secretly knows that φ" in the sense that (1) a knows φ (φ is true in all the states she considers possible) and (2) that a knows that b doesn't know $\varphi^\star$: $M, w \models p$ iff $w \in V(p)$; $M, w \models \neg\varphi$ iff $M, w \not\models \varphi$; $M, w \models \varphi \wedge \psi$ iff $M, w \models \varphi$ and $M, w \models \psi$; $M, w \models \Box\varphi$ iff $\forall w' \in W$, if wR_aw' then $M, w' \models \varphi$ and $\exists u \in W$ s.t. $w'R_bu$ and $M, u \models \neg\varphi$.

This thus interprets the box as a combined knowledge and ignorance modality, and it should come as no surprise that it is not normal – it is not monotonic. Perhaps more surprisingly, it is captured exactly by the Interpolation rule (in addition to the adjunctive axiom (C) and the (S) axiom).

Theorem 4. **ICS** *is sound and complete** *with respect to the class of all Kripke models (interpreting the modal box as secretly-knowing).*

*Or *believing*. We use "knowing" in the loose sense of [2] here.

*This generalises to any finite set of agents but we consider the simplest case of two agents in this abstract for brevity.

*This of course does not capture all aspects of secretly-knowing – see [5] for discussion.

*Sound and complete: contains a formula iff it is valid (true inn all states in all models).

A common distinction between *knowledge* and*belief* is that for the former, reflexivity of accessibility is often assumed. We get a corresponding result also in that case, by adding axiom (T): $\Box\varphi \to \varphi$.

Theorem 5. **ICST** *is sound and complete with respect to the class of all reflexive Kripke models.*

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Does DC imply AC_ω, uniformly?

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SPEAKER: Alessandro Andretta.

$\mathsf{AC}_\omega(X)$ asserts that given $\emptyset \neq A_n \subseteq X$ there are $a_n \in A_n$ for all $n \in \omega$, while $\mathsf{DC}(X)$ asserts that given $R \supseteq X \times X$ such that $\forall x \exists y (x \; R \; y)$ there are $(a_n)_{n\in\omega}$ such that $a_n \; R \; a_{n+1}$ for all $n \in \omega$. It is well known that DC, that is $\forall X \, \mathsf{DC}(X)$, implies AC_ω, that is $\forall X \, \mathsf{AC}_\omega(X)$. But does this implication holds uniformly? In other words: does $\mathsf{DC}(X) \Rightarrow \mathsf{AC}_\omega(X)$ for all X? Clearly there are many sets X for which this implication is valid, for example $X = \mathbb{R}$.

Working in ZF we prove the following.

Theorem 1. *Assume* $\mathsf{AC}_\omega(\mathbb{R})$*. Then* $\mathsf{DC}(X) \Rightarrow \mathsf{AC}_\omega(X)$*, for all* X*.*

Theorem 2. *It is consistent with* ZF *that there is a set* $A \subseteq \mathbb{R}$ *such that* $\mathsf{DC}(A)$ *holds and* $\mathsf{AC}_\omega(A)$ *fails.*

Theorem 3. $\mathsf{DC}(X) \wedge \mathsf{DC}(Y) \Rightarrow \mathsf{DC}(X \cup Y)$*, for all* X, Y*.*

Psychoanalytic theory and logic

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SPEAKER: Giulia Battilotti.

Federigo Enriques, at the beginning of the last century, argued that *Logic can be regarded as a part of Psychology* ([1], p.164). Our research develops a formalized approach to the foundations of psychoanalysis in logic, by considering a pre-logical setting, interpreting and integrating some views developed in psychoanalysis by Freud, Matte Blanco, Bion and other authors. In turn, the pre-logical elements can be read as seeds for logic.

We first consider quantified formulae on non-extensional domains termed infinite singletons [10], that corresponds to the thing-presentations of objects, a concept at the basis of Freudian theorization, see [2], 1891. Thing presentations are non-verbal open representations of objects operated by the Unconscious, which can access consciousness only when closed by words. The model of infinite singletons can grasp the logical features of the structural Unconscious, a negationless environment of infinite mental objects, characterized by Freud in The Interpretation of Dreams [3], then logically interpreted by Matte Blanco in [6].

Then we consider the modal operator of S4, derived by abstracting with respect to the collection of infinite singletons which refer to the same objects, and characterizable by an infinite singleton itself. It can attribute a sharp yet undefined value to an object, namely is situated in between open and closed presentations [11]. Further, one can assume that some form of "priming" for the process of abstraction can be considered, as in [7]. A set of different independent possibilities for priming are characterized [12], as a basis for our conscious judgements from their Unconscious origins. Beyond a neuter form, that corresponds to no priming, one has a positive, a negative and an impossible priming, the last associated to the failure of thing-presentations. The two others are the assertive and the negative finite views of the original infinite neuter modality. In logic, one can see that the negative view of the modality, first described in [5], gives rise to negation and non contradiction, and that the original impossible modality is overcome when expressed by the dual of necessity, namely possibility. The last confines singletons to their usual role and makes the usual finite and infinite sets possible. The modal operators, then, are the key to shift from the mode of the Unconscious to rational thinking, a view consistent with the the moving from the First to the Second Topic in Freud, introducing a normative instance moderating the encounter of the psychic dimension with the external reality, see [4].

The method adopted to introduce the logical objects we need is taken from Basic Logic [9], considering its reflection principle and its symmetry theorem. The model has been originally developed in physics, as a model of quantum states, conceived as infinite singletons; the modal operator of S4 is then introduced as an abstract spin projector.

Overall, our proposal mirrors Gödel's idea that finiteness leads to incompleteness, and that completeness requires the integration of means not yet included in formal logic. In particular we think that our proposal reinterprets in terms of qubits Gödel's modal interpretation of intuitionistic logic, [8], that is infinite-valued, as he also proved. In it, intuitionistic logic is produced by addition, the addition of the modality to classical propositional logic, namely the logic of bits. With qubits, a logical environment, preserving an infinite aspect, can be, rather, carved out, from the infinite modal pre-logic, in accordance with Matte Blanco's idea that the primary mode of sets, that is of the objects of the mind, is infinite and not finite, and that bivalence is present only in consciousness.

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Default quantifiers: an algebraic investigation

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SPEAKER: Sofia Becatti.

In [2] K. Schlechta proposed a predicate calculus based on the idea of *default quantifier*, which we will denote by ∇. Intuitively, its function is to describe when a certain property holds in the majority of cases; in other words $\nabla x P(x)$ can be interpreted as "for the majority of x, $P(x)$ holds". The aim of Schlechta is formalizing what he calls "default reasoning"; this means that a property is true when it holds most of the time, according to our experience.

The language considered by Schlechta is any first order language endowed with the symbol ∇ and the well-formulas are all the ones of first order logic together with all the formulas in the form $\nabla x \phi(x)$ where ϕ is itself a well-formed formula an x is a variable. In order to define the concept of model of such a language, Schlechta encounters the necessity of formalizing what is meant by majority of cases; this is fulfilled introducing the concept of N-system on a given set. Intuitively, an N-system on a set can be seen as a collection of big subset of the chosen universe; to accomplish this purpose, Schlechta lists three properties that subsets in the N-system need to satisfy in order to be considered big: supersets of big sets should be big, the universe should be a big set itself and finally two big sets should not have empty intersection.

The concept of N-system is strictly connected to the one of model of the above language, which is defined as a model M of the language without ∇ (according to the usual definition of model of a language) together with an N-system N on M. Clearly, the notion of validity for formulas not containing ∇ is defined in the canonical way, while a formula of the form $\nabla x \phi(x)$ is said to be valid in the model if there exists a set $A \in N$ such that $\phi(a)$ is a valid formula for every $a \in A$.

After that, Schlechta introduces an axiomatization for his predicate calculus endowing any axiomatization of first order logic with new axiom schemata and then he proves some soundness and completeness results.

Our aim is to describe an algebraic counterpart of this predicate calculus in the same fashion as monadic algebras are for classical predicate calculus. The natural way to proceed is to follow the same path as in the classical paper by Halmos [1]; without entering into details, it is possible to define a variety of algebras (which will be of course the variety of Boolean algebras with one additional unary operation) that is exactly the algebraic counterpart of the monadic fragment of a fragment of Schlechta's predicate calculus. In fact, Schlechta's calculus contains not only the default quantifier but also the classical quantifiers; we have decided to consider only the default quantifier in order to avoid unnecessary complications.

In particular we define a **Boolean algebra with default quantifier** as a Boolean algebra D endowed with a unary operator ∇ satisfying the following properties:

1. $\nabla 1 = 1$;

2. $\nabla(x \vee y) \geq \nabla x \vee \nabla y$;

3. $\nabla x \wedge \nabla \neg x = 0$;

4. $\nabla(x \wedge \nabla y) = \nabla x \wedge \nabla y$;

5. $\nabla(x \vee \nabla y) = \nabla x \vee \nabla y$.

From the definition it is clear that Boolean algebras with default quantifier form a variety, which we denote by D.

After having introduced such a variety, we focus on the study of some algebraic properties of Boolean algebras with default quantifier; in particular we firstly prove some results about the nature of the image of the default quantifier ∇, which turns out to be a Boolean subalgebra of D, and then we investigate how ∇ acts on each element of the algebra, once fixed the subalgebra C that is its image. We'll prove that the action of ∇ on each element is closely related to the nature of the set $N = \{x \in D : \nabla x = 1\}$: specifically, we look for necessary and sufficient conditions for compatibility between N and C, where by compatibility we mean that it is actually possible to define a default quantifier ∇ on D such that N is the set of elements whose ∇ is equal to 1 and C is the image of ∇. Finally we show that the action of ∇ on each element of D is uniquely determined once fixed C and N in such a way that they are compatible.

After that we focus on the study of the congruence lattice $Con(D)$ of a generic Boolean algebra with default quantifier D; if we denote by $Fil(D)$ the lattice of the filters of D which are also closed under ∇, it turns out that $Con(D)$ is isomorphic to a sublattice of $Fil(D)$. This allows us to investigate in which cases D is simple or subdirectly irreducible; such an analysis results in two theorems containing necessary and sufficient conditions for a Boolean algebra with default quantifier to be either simple or subdirectly irreducible but not simple. Those results lead to some straightforward consequences: we notice that there are Boolean algebras with default quantifier which are subdirectly irreducible but not simple, hence the variety D is not semisimple; furthermore we provide an example of simple Boolean algebra with default quantifier that has a non-simple subalgebra, allowing us to state that D does not have the CEP property.

Moreover, we study the structure of the initial part of the lattice of subvarieties of D; this will be done through several applications of Jónsson's Lemma in its finite version: we start by studying the varieties generated by the smallest subdirectly irreducible Boolean algebras with default quantifier and then we'll proceed moving on to bigger ones. Clearly, the variety B of Boolean algebras will be a subvariety of D.

It is also worth to notice that the default quantifier ∇ can be seen as a modal operator, where the modality is interpreted as "valid in the most of cases". It turns out that the modal logic that the default quantifier gives rise to is not a normal one; it could therefore be interesting to study its neighborhood semantics and investigate how it is related to the modal logic systems that are already known.

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On the structure of balanced residuated partially-ordered monoids

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SPEAKER: Stefano Bonzio.

Based on a joint work with J. Gil-Férez, P. Jipsen, A. Prenosil and M. Sugimoto

A *residuated poset* is a structure $A, \leq, \cdot, , , 1$ where $A, \leq$ is a poset and $A, \cdot, 1$ is a monoid such that the residuation law $x \cdot y \leq z \iff x \leq z/y \iff y \leq x\backslash z$ holds. A residuated poset is *balanced* if it satisfies the identity $x\backslash x \approx x/x$. By generalizing the well-known construction of Płonka sums, we show that a specific class of balanced residuated posets can be decomposed into such a sum indexed by the set of positive idempotent elements. Conversely, given a semilattice direct system of residuated posets equipped with two families of maps (instead of one, as in the usual case), we construct a residuated poset based on the disjoint union of their domains. We apply this approach to provide a structural description of some varieties of residuated lattices and relation algebras.

Ordinal Analysis of Well-Ordering Principles

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SPEAKER: Gabriele Buriola.

Well orders play a crucial role in many areas of mathematics, not last mathematical logic; therefore, closure properties for well orders deserve a special attention. In this view, a relevant topic is given by the study of Well-Ordering Principles, WOP's. Generally speaking, denoting with Ω the set of countable ordinals and with WO the property of being a well-order, given an ordinal function $g : \Omega \to \Omega$ the corresponding Well-Ordering Principle, WOP(g), amounts to:

$$\forall \mathfrak{X}\, [\mathrm{WO}(\mathfrak{X}) \to \mathrm{WO}(g(\mathfrak{X}))],$$

namely WOP(g) asserts that g preserves well orderedness.

Many WOP's have been considered in reverse mathematics. Using lambda notation, i.e. $\lambda\mathfrak{X}.\alpha(\mathfrak{X})$ denotes the ordinal function sending $\mathfrak{X}$ to $\alpha(\mathfrak{X})$ (with $\alpha(\mathfrak{X})$ an ordinal term containing $\mathfrak{X}$), the following equivalences hold:

Theorem 1. *Over RCA_0:*

- *ACA_0 is equivalent to WOP($\lambda\mathfrak{X}.\omega^{\mathfrak{X}}$) [Girard [3]];*

- *ACA_0^+ is equivalent to WOP($\lambda\mathfrak{X}.\varepsilon_{\mathfrak{X}}$) [Marcone and Montalbán [5]];*
- *ATR_0 is equivalent to WOP($\lambda\mathfrak{X}.\varphi\mathfrak{X}0$) [Friedman, Montalbán and Weiermann];*
- *ATR_0^+ is equivalent to WOP($\lambda\mathfrak{X}.\Gamma_{\mathfrak{X}}$) [Rathjen [6]].*

Lately Arai [1, 2] studied the general behaviour of WOP's; for a normal function g, namely an ordinal function which is strictly increasing and continuous, and its derivative g', i.e. the ordinal function enumerating the fixed points of g, Arai gave the following characterizations.

Theorem 2 (Arai [2]). $|ACA_0 + WOP(g)| = g'(0) = \min\{\alpha \mid g(\alpha) = \alpha\}$.

Theorem 3 (Arai [2]). *Over ACA_0, the following are equivalent:*

- *$WOP(g')$;*
- *$(WOP(g))^+$;*

where $(WOP(g))^+$ means that every set is contained in a countable coded ω-model of ACA_0 + $WOP(g)$, see [2, Definition 2] for a detailed definition.

We have recently extended theor:Arai [3] to a larger class of ordinal functions. More precisely, we dropped the normality condition for g requiring only the following two hypotheses:

1. g is weakly increasing, i.e. $\alpha \leqslant \beta \;\Rightarrow\; g(\alpha) \leqslant g(\beta)$;
2. $g'(0)$ is an epsilon number, i.e. $\omega^{g'(0)} = g'(0)$.

Since g does not need to be a normal function, and therefore may lack fixed points, by $g'(0)$ we denote the first ordinal closed under g, namely $g'(0) := \min\{\alpha > 0 \mid \forall\beta < \alpha\; g(\beta) < g(\alpha)\}$.

As an application of this novel extension, we compute the proof-theoretic ordinals of the following two WOP's:

- $|\mathrm{ACA}_0 + \mathrm{WOP}(\lambda\mathfrak{X}.\vartheta(\Omega^\omega \cdot \mathfrak{X})| = \vartheta(\Omega^{\omega+1})$;
- $|\mathrm{ACA}_0 + \mathrm{WOP}(\lambda\mathfrak{X}.\sup_n \vartheta(\Omega^n \cdot \mathfrak{X})| = \vartheta(\Omega^\omega + \omega)$;

where ϑ is a so-called collapsing function [7].

The two previous WOP's stem from the ordinal analysis of two different versions of Kruskal's theorem, a celebrated result in the theory of well quasi-orders. More precisely, for a wqo Q let $T(Q)$ be the set of finite trees with labels in Q and $T^n(Q)$ be the set of finite trees with labels in Q and branching degree less or equal to n (i.e. every node in a tree has at most n successors); then, denoting with $\mathrm{KT}_\ell(\omega)$ standard Kruskal's theorem ("if Q is wqo then $T(Q)$ is a wqo") and with $\mathrm{KT}_\ell(n)$ the bounded version ("if Q is wqo then $T^n(Q)$ is a wqo"), the following equivalences hold [3]:

- $\mathrm{ACA}_0 \vdash \mathrm{KT}_\ell(\omega) \longleftrightarrow \mathrm{WOP}(\lambda\mathfrak{X}.\vartheta(\Omega^\omega \cdot \mathfrak{X}))$;
- $\mathrm{ACA}_0 \vdash \forall n\, \mathrm{KT}_\ell(n) \longleftrightarrow \mathrm{WOP}(\lambda\mathfrak{X}.\sup_n \vartheta(\Omega^n \cdot \mathfrak{X}))$.

These equivalences, combined with the previous computations regarding proof-theoretic ordinals for WOP's, allow to easily calculate the proof-theoretic ordinals of $\mathrm{KT}_\ell(\omega)$ and $\forall n\,\mathrm{KT}_\ell(n)$. We briefly recall that the unlabelled case was treated by Rathjen and Weiermann [7] who obtained the following result:

$$\mathrm{ACA}_0 \vdash \mathrm{KT}(\omega) \longleftrightarrow \forall n\,\mathrm{KT}(n) \longleftrightarrow \vartheta(\Omega^\omega)\,.$$

In the same spirit, we studied the well-ordering principles and the proof-theoretic ordinals corresponding to the following closure properties for well quasi-orders:

- $\forall Q, \forall n\,[Q\,\mathrm{wqo} \to Q^n\,\mathrm{wqo}]$, with $Q^n := Q \times \cdots \times Q$;
- $\forall Q, \forall n\,[Q\,\mathrm{wqo} \to n{\cdot}Q\,\mathrm{wqo}]$, with $n{\cdot}Q := Q + \cdots + Q$.

The main goal of our research is double:

- compute the proof-theoretic ordinals of relevant well-ordering principles;
- draw a thorough picture, in term of WOP's, of the most common closure properties for well quasi-orders.

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A calculus for modal compact Hausdorff spaces

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SPEAKER: Luca Carai.

Dualities between algebras and topological spaces provide a crucial tool in the study of logics, algebras, and topologies. The groundbreaking work of Stone established the duality between Boolean algebras and Stone spaces, paving the way for numerous subsequent studies on dualities. Among the generalizations of Stone duality, there is *de Vries duality* [4] between compact Hausdorff spaces and de Vries algebras, which are complete Boolean algebras enriched with a binary relation satisfying some specific properties. From a logical perspective, de Vries algebras have been studied in [2, 5], where the *strict symmetric implication calculus* $\mathsf{S^2IC}$ is defined by extending the classical propositional calculus with a binary connective called strict implication, and it is shown that $\mathsf{S^2IC}$ is strongly sound and complete with respect to de Vries algebras. This yields that $\mathsf{S^2IC}$ is also sound and complete with respect to compact Hausdorff spaces.

Modal spaces are Stone spaces endowed with a relation R satisfying some 'continuity' conditions. These spaces play a fundamental role in the study of modal algebras, also known as Boolean algebras with operators, because Stone duality generalizes to Jónsson-Tarski duality between modal spaces and modal algebras. In [1] *modal compact Hausdorff spaces* are introduced as the compact Hausdorff generalization of modal spaces:

Definition 1. A *modal compact Hausdorff space* is a pair (X, R) consisting of a compact Hausdorff space X and a binary relation R on X such that

1. $R[x]$ is closed for each $x \in X$;
2. $R^{-1}[F]$ is closed for each closed $F \subseteq X$;
3. $R^{-1}[U]$ is open for each open $U \subseteq X$.

In [1] it is proved that de Vries duality generalizes to a duality between modal compact Hausdorff spaces and upper continuous modal de Vries algebras. Developing a sound and complete calculus for modal compact Hausdorff spaces was left as an open problem in [1].

We solve this problem by introducing the calculus $\mathsf{MS^2IC}$. This calculus is obtained by extending the strict symmetric implication calculus $\mathsf{S^2IC}$ with a modal operator $\Box$ and adding to $\mathsf{S^2IC}$ specific axioms and a Π_2-rule that expresses upper continuity.

Theorem 2. $\mathsf{MS^2IC}$ *is strongly sound and complete with respect to the class of modal compact Hausdorff spaces.*

Π_2-rules are non-standard inference rules that naturally possess $\forall\exists$ counterparts, and played an important role in the axiomatization of de Vries algebras using the strict symmetric implication calculus $\mathsf{S^2IC}$ in [2, 5]. However, in the same works it was shown that these rules were, in fact, *admissible* in the calculus, and hence they could be omitted in the axiomatization of $\mathsf{S^2IC}$. This observation generates the question of whether Π_2-rules are indeed necessary for the axiomatization of $\mathsf{MS^2IC}$.

We show that the Π_2-rule expressing upper continuity is admissible in $\mathsf{MS^2IC}$. Our admissibility proof deviates from the general methods introduced in [2, 5] as our framework lacks the amalgamation/interpolation properties, essential in those methods. Instead, we obtain the proof by first developing a relational semantics for $\mathsf{MS^2IC}$ and subsequently applying bisimulation expansions.

The Kripke frames providing the relational semantics for $\mathsf{MS^2IC}$ are triples (X, T, S) consisting of a set X together with a ternary relation T and a binary relation S. The relation T is used to interpret the strict implication, while S interprets the modality $\Box$. The canonicity of the axioms of $\mathsf{MS^2IC}$ allows us to establish the following.

Theorem 3. $\mathsf{MS^2IC}$ *is Kripke complete.*

The results presented in this talk can be found in the preprint [3].

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Continuous reducibility is a well-quasi-order on continuous functions with analytic 0-dimensional domain

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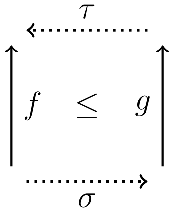
SPEAKER: Raphaël Carroy.

Definition 1. Given topological spaces X, X', Y, Y', we say that a function $f : X \to Y$ *continuously reduces* to a function $g : X' \to Y'$ if there are two continuous functions $\sigma : X \to X'$ and $\tau : \mathrm{im}(g \circ \sigma) \to \mathrm{im}(f)$ such that $f = \tau \circ g \circ \sigma$. We also say that the pair (σ, τ) *continuously reduces*, or simply *reduces*, f to g.



Continuous reducibility is a transitive and reflexive relation, which makes it a *quasi-order*. As usual with quasi-orders, there is an induced equivalence relation: we write $f \equiv g$ when both $f \leq g$ and $g \leq f$ hold, and we say that f and g are *continuously equivalent*. We also denote by $f < g$ the relation $f \leq g$ and $f \not\geq g$, and we say that f and g are *incomparable* when both $f \not\leq g$ and $g \not\leq f$.

An *antichain* is a set of pairwise incomparable elements, and an *infinite strictly descending chain* is a sequence $(f_n)_{n \in \mathbb{N}}$ satisfying $f_{n+1} < f_n$ for all $n \in \mathbb{N}$. A quasi-order is called a *well-quasi-order* if it has no infinite antichains and no infinite descending chains.

A topological space is *Polish* if it is separable and completely metrizable, and *zero-dimensional* if it has a basis consisting of *clopen sets*, that is sets that are both open and closed. A topological space is *analytic* if it is a continuous image of a Polish space.

Our main result is the following theorem.

Continuous reducibility is a well-quasi-order on the class of continuous functions from an analytic zero-dimensional space to a separable metrizable space.

The analyticity hypothesis in Theorem is only used to show that all such continuous functions with uncountable image are equivalent to the identity function on the Cantor space. In fact, we prove

Continuous reducibility is a well-quasi-order on the class of continuous functions from a separable metrizable zero-dimensional space to a countable metrizable space.

The above theorem is proved following a dichotomy which is reminiscent of the case of spaces. Recall that a space is *scattered* if any of its non-empty subsets contains an isolated point. We suggest to make the following definition: a function f between topological spaces is *scattered* if any non-empty subset of its domain contains an open set on which f is constant. Every continuous function with a scattered image is scattered, but there are scattered continuous functions whose image (take for example any bijection $: \mathbb{N} \to \mathbb{Q}$) or domain (a constant function on $\mathbb{Q}$) is not scattered. While the space $\mathbb{Q}$ of rationals is universal for countable metrizable spaces, any non scattered metrizable space contains a copy of $\mathbb{Q}$. We show analogous results for functions with $\mathrm{id}_{\mathbb{Q}}$ in the role of $\mathbb{Q}$, thereby establishing that, up to continuous equivalence, $\mathrm{id}_{\mathbb{Q}}$ is the only non scattered continuous function from a metric space to a countable metric space.

With this in mind, the main challenge in proving Theorem concerns the scattered functions. For the purpose of this paper, we therefore introduce the class of continuous functions $f : X \to Y$ such that f is scattered, X is separable zero-dimensional metrizable and Y is countable metrizable. The result at the heart of our main theorem can then be stated as follows.

Continuous reducibility is a well-quasi-order on the class .

We thus in particular answer positively [1, Question 5.5], as then conjectured.

This is a joint work with Yann Pequignot.

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Strong truth classes via approximations

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SPEAKER: Cesary Cieśliński.

Introduction

For a model M of Peano arithmetic (PA), a truth class is a subset X of M-sentences⋆ which makes true all Tarski-style compositional truth axioms.⋆

It is known that a truth class can be constructed for an arbitrary countable recursively saturated model of PA. However, the original proof of this theorem (see [3]) using the so-called 'technique of approximations' was difficult to follow for many readers, with the machinery of approximations being one of the main stumbling blocks. Accordingly, the question has been asked whether the result can be proved by purely classical methods. One successful attempt in this direction can be found in [2], where classical techniques of formal semantics are employed.

On the other hand, in [1] the theorem was proved by the classical techniques of proof theory, namely, by cut elimination. Coupled with Enayat and Visser's construction, this makes the field of truth classes accessible to the logicians whose primary interest is either model theory or proof theory.

However, it was not obvious how to generalize the techniques from [1] in order to permit construction of stronger truth classes. In the proposed talk we fill in this gap. Our

⋆An M-sentence is an objects $a \in M$ such that $M \models Sent(a)$, where $Sent(a)$ is an arithmetical predicate expressing that a is an arithmetical sentence.

⋆Thus, for example, we would have: $(M, X) \models \forall\varphi, \psi T(\varphi \land \psi) \equiv T(\varphi) \land T(\psi)$, with the predicate "$T(x)''$ interpreted by X.

main objective is to present a construction of a strong truth class, containing all instances of schemata derivable in Peano arithmetic. In our proof the notion of approximation plays a crucial role; however, unlike in [3], we develop approximations as a purely proof-theoretic technique, easily accessible to researches versed in classical proof theory. In this way, the versatility of proof-theoretic approach to truth is vindicated.

Outline

We work with a fixed countable and recursively saturated model M of PA. We start by presenting a proof system called 'M-logic' (ML in short). Intuitively, ML permits us to process arbitrary sentences in the sense of M, including the nonstandard ones. The system is described externally (not in the model) in the form of a classical sequent calculus. We use the notation '$\Gamma \Rightarrow \Delta$' when referring to sequents. We shall always assume that both Γ and Δ are externally finite sequences of M-sentences. Unlike in Gentzen's original system, we do not admit formulas with free variables in the sequents. This deficiency is compensated by the presence of infinitary rules for quantifiers in ML.* In addition, ML contains truth rules, which guarantee that every true literal (atomic arithmetical sentence or its negation) is a theorem of ML.

Proofs in ML are trees of finite height, where the height of a proof is the length of its maximal path. By definition, trees with no maximal path do not qualify as proofs in ML.

The proof of the following lemma can be found in [1].

Lemma 1. *Let $M \models I\Delta_0 + \exp$ be countable and recursively saturated. If ML is consistent, then there is a truth class in M, which contains all theorems of ML.*

It is important that the lemma remains true when ML is enriched with additional initial sequents (axioms). Now we introduce a notion of a schema.

Definition 2 (Schema Template)**.** An $\mathcal{L}$ *template* (for a schema) is given by an $\mathcal{L} \cup \{P\}$-sentence $\tau(P)$ obtained by augmenting $\mathcal{L}$ with a n-ary predicate $P(x_1, \dots, x_n)$. An $\mathcal{L}$-sentence is an *instance* of τ if it is of the form $\forall v\tau[\psi(t_1, \dots, t_n, v)/P]$, where $\tau[\psi(t_1, \dots, t_n, v)/P]$ is the result of substituting all subformulae of the form $P(t_1, \dots, t_n)$ with $\psi(t_1, \dots, t_n, v)$, renaming bound variables as necessary so to avoid clashes.

Given a L_{PA}-template τ, let $Sch(PA) := \{\tau \mid$ proves all L_{PA}-instances of $\tau\}$.

Definition 3. For $\tau_1 \dots \tau_n \in Sch(PA)$, we define the system $MLS^{\tau_1 \cdots \tau_n}$ as ML with additional initial sequents of the form $\emptyset \Rightarrow \varphi$, such that φ is an instance of one of the indicated schemas.

Our **main result** is that for all $\tau_1 \dots \tau_n \in Sch(PA)$, $MLS^{\tau_1 \cdots \tau_n}$ is consistent. By Lemma 1 it follows that recursively saturated models expand to models with a truth class containing all substitutions of finitely many schemata. Then an easy compactness argument shows that a truth theory containing Tarskian compositional axiom, enriched with infinitely many axioms of the form 'For every ψ, if ψ is an instance of τ, then $T(\psi)$' (for every $\tau \in Sch(PA)$) is a conservative extension of PA.

*For example, the rule $\exists$-left has the form: from the infinite set of assumptions $\{\varphi(a), \Gamma \Rightarrow \Delta : a \in M\}$, infer: $\exists x\varphi(x), \Gamma \Rightarrow \Delta$.

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Fibrations, quotient completions and descents in foundations of constructive mathematics

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SPEAKER: Cipriano Junior Cioffo.

Categorical tools are indispensable to define models of logical systems formulated in the field of type theory. In particular, Benabou's fibrations and indexed categories (see [1]) had been successfully employed to model dependent type theories used as foundations of constructive mathematics, such as Coquand's Calculus of Constructions (**CoC**), Martin-Löf's type theory (MLTT) and the more recent Voevodsky's Homotopy Type Theory (**HoTT**). The appealing aspect of type-theoretic foundations is the possibility to be formalized in proof assistants and perform computer verification of proofs.

In this talk we address the problem of modelling the two-level structure of the *Minimalist Foundation* (MF), ideated by the second author and Sambin in [10] and fully formalized in [5]. Indeed, MF offers a predicative constructive mathematical environment with minimal assumptions, which is compatible with most relevant constructive foundations including **CoC**, **MLTT**, **HoTT**, Aczel's CZF and the general theory of elementary toposes.

And, such a minimality of **MF** was made possible by structuring **MF** as a two-level structure consisting of an *intensional level* mTT formulated as a dependent type theory suitable for computer formalization, an *extensional level* emTT formulated as a many-sorted logic closer to the traditional mathematics language, and an interpretation of emTT in mTT through a *quotient model* (see [5]).

The long term goal of our research would be to export such as a minimality from the formal syntactic level to a model theoretic one.

While for interpreting both levels of **MF** we can adapt fibrational tools used to model MLTT thanks to results in [4], it is still an open problem to describe categorically the interpretation of the extensional level $\mathbf{emTT}$ in the quotient model of so called “setoids" within the intensional level $\mathbf{mTT}$ in [5].

More in detail, while the whole quotient model of **MF** can be described categorically as a suitable elementary quotient completion introduced in [9], it is an open problem how to describe the structure of *dependent setoids a' la Bishop* used to model set families of $\mathbf{emTT}$ in $\mathbf{mTT}$. The problem boils down to lift the indexed categories interpreting the various types of **mTT** as a structure of the codomain fibration of the elementary quotient completion describing the quotient model of **MF**.

In this talk, we will discuss how to obtain such a categorical description using the descent theory in [2, 3] so that we can fully enrich an elementary quotient completion built out of a fibrational model **mTT** into a fibrational model of **emTT**.

Other relevant examples of such fibrations obtained using descents can be built both out of the *predicative version of Hyland's Effective topos* **Eff** in [6] (within Feferman's theory of non-iterative fixpoints) and a *constructive and predicative version of* **Eff** with **CZF+ REA** based on the work in [6] and [8].

In this way we reach a categorical description of how *proofs* in **MF** can be turn into *programs*.

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Learning isomorphism problems with countably many isomorphism types

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SPEAKER: Vittorio Cipriani

This talk combines ideas from computable structure theory inductive inference and descriptive set theory to study learning of families of structures. The framework we use was defined in a series of papers by Bazhenov, Fokina, Kötzing, and San Mauro, and models the following scenario: Given a family of structures $\mathfrak{K}$, a *learner* receives more and more information about the atomic diagram of a copy of some $\mathcal{A} \in \mathfrak{K}$ and, at each stage, is required to output a conjecture about the isomorphism type of such a structure. We say that $\mathfrak{K}$ is $\mathbf{Ex}$-*learnable* if there exists a learner that stabilizes to the correct conjecture after finitely many steps.

Recently, together with Bazhenov and San Mauro, we gave a descriptive set-theoretic characterization of $\mathbf{Ex}$-learning. Namely, we showed that a family of structures is $\mathbf{Ex}$-learnable if and only if the corresponding isomorphism problem continuously reduces to E_0, the equivalence relation of eventual agreement on infinite binary sequences. Replacing E_0 with other equivalence relations, one obtains a hierarchy to rank such isomorphism problems. That is, a family of structures $\mathfrak{K}$ is E*-learnable*, for an equivalence relation E, if there is a continuous reduction from the isomorphism problem associated with $\mathfrak{K}$ to E.

To get a better understanding of when a family of structures is E-learnable it is useful to provide a model-theoretic characterization of E-learnability. Some characterizations are already present in the literature. Here we show that a family of structures $\mathfrak{K}$ is E-learnable for E being the (iteration of the) Friedman-Stanley jump of equality on natural numbers or on Cantor space if and only if the inclusion relation of the Σ_n^{inf}-theories is a partial order on $\mathfrak{K}$. We also show that other learning criteria coming from the classical setting of inductive inference of formal languages or recursive functions have a nice model-theoretic characterization.

This talk collects joint works with Bazhenov, Jain, Marcone, San Mauro and Stephan.

On the Compatibility of Constructive Predicative Theories with Weyl's Classical Predicativity

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SPEAKER: Michele Contente

It is well known that most foundations for Bishop's constructive mathematics are not compatible with a classical predicative development of analysis as put forward by Weyl in *Das Kontinuum* [8]. Among the most popular systems proposed as foundations for Bishop's constructive mathematics there are Aczel's constructive set theory **CZF** and Martin-Löf Type Theory **MLTT** [4]. More recently, a development of the latter that is known as Homotopy Type Theory **HoTT** [7] has gained much attention, especially for its applications to synthetic homotopy theory and higher categories.
We first discuss why these systems are not compatible with classical predicative theories. Indeed, when they are extended with classical logic, they become impredicative. Hence, they cannot be equiconsistent with their classical counterparts.
Our analysis will highlight the role played by the *rule of unique choice* in the derivation of these incompatibility results. We will show that already the weaker system of Heyting Arithmetic in all finite types when extended with the law of excluded middle and a *weak* form of the rule of unique choice, is enough strong to interpret second-order Peano arithmetic, that is an impredicative theory. Then this result can be extended to the more powerful systems mentioned above.
We argue that a possible way-out could be provided by the two-level system known as Minimalist Foundation **MF** [3, 5].

MF includes exponentiation for function as λ-terms, while functional relations do not necessarily form a set. Furthermore, choice principles are not internally valid in **MF** [2]. We show that even the (weak) rule of unique choice, which identifies functional relations with functions as λ-terms, is not valid in **MF**. This restriction suggests the need for a *point-free* development of topology, as advocated by Martin-Löf and Sambin [6], and analysis [5]. In this perspective, we will assess the status of various constructions of real numbers in **MF** in comparison to those in other constructive systems such as **CZF**, **MLTT** and **HoTT**. Ultimately, we will contend that **MF** promises to be a natural crossroads between Bishop's constructive mathematics and Weyl's classical predicativity.

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Arbitrary Frege Arithmetic

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SPEAKER: Ludovica Conti

Abstractionist theories in philosophy of mathematics are systems composed by a logical theory augmented with an abstraction principle (AP), of form: $\forall X \forall Y(@X = @Y) \leftrightarrow E(X,Y)^{\star}$ – that introduces, namely rules and implicitly defines, a term-forming operator @ by means of an equivalence relation E. As is well-known, the seminal abstractionist program, Frege's Logicism, failed*: Russell's Paradox proved its inconsistency and, *a fortiori*, its non-logicality. In the last century, both the issue of consistency and the issue of logicality have been resumed in the abstractionist debate (cf. [12], [7], [1], [4], [3]). More precisely, on the one side, different revisions of Frege's original system have been proposed in order to avoid Russell's Paradox and to obtain a consistent system that is strong enough to derive (at least, relevant portion of) Peano Arithmetic. On the other side, given a semantical definition of logicality as permutation invariance, some abstraction principles have been proved to be logical ([1], [4]).

Nevertheless, many concerns are still open. Particularly, regarding the preliminary condition of consistency, the ways out of Russell's Paradox proposed so far does not precisely mirror a corresponding explanation of the origin of the contradiction and often imply a weakening of the hoped strength of the theory (cf. [11], [13], [6])*; regarding the

*In the rest of the paper, I'll adopt this axiomatic version of AP. Given full Comprehension Axiom Schema (that will be assumed in the systems that we'll investigate), it is equivalent to the schematic form: $@x.\alpha(x) = @x.\beta(x) \leftrightarrow E(\alpha(x), \beta(x))$.

*It was proposed with the foundational purpose to derive arithmetical laws as logical theorems and to define arithmetical expressions by logical terms.

*In [5] and [2], second-order Peano Axioms are recovered but by appealing to stronger logical resources – i.e. double-sorted variables issue of logicality, an undesired dilemma overshadows the abovementioned results: precisely in case of logical (i.e. permutation invariant) abstraction principles, their implicit *definienda* turn out to be non logical ([1]) – so preventing a full achievement of Logicist goal.

My preliminary aim consists in arguing that these – apparent unrelated –problems have a common source in some unquestioned assumptions of Freges' project (inherited also by the following abstractionist programs). I argue that such assumptions are part of what we can call the Traditional view of abstraction, that includes the choice of classical logic as the base theory, with the related semantical consequence of full referentiality of the vocabulary, and the choice of a so-called Canonical interpretation function for all the (both primitive and defined) expressions of the language.

In the rest of the talk, I show that by renouncing to one or both of these problematic assumptions we can recover consistency and/or logicality. More precisely, I propose a double revision of Frege's Logicist program: on the one side, weakening Canonical interpretation function for the implicitly defined (abstract) expressions of the vocabulary (cf. [3]), I prove that any consistent revision of BLV turns out to be logical (i.e. permutation invariant); on the other side, I show that such an arbitrary interpretation, on a (negative) free logic background, allows us to identify a restriction of BLV, able to precisely exclude the paradoxical concepts, namely to avoid Russell's Paradox, but, at the same time, to preserve the derivational strength necessary to derive second-order Peano axioms. This means that this system – that we'll call Arbitrary Logicism, precisely renouncing to the Traditional assumptions mentioned above, is able to recover both Frege's goals of consistency and logicality.

The logical part of the language of Arbitrary Logicism, L_F, includes denumerably many first-order variables $(x, y, z, ...)$, denumerably many second-order variables $(X, Y, Z, ...)$, logical connectives (¬, →) and a first-order existential quantifier (∃)⋆. We can also usefully define a predicative monadic constant (E!), whose extension is equal to the range of identity: E!a $=_{def} \exists x(x = a)$. The only non-logical primitive symbol is the term-forming operator ϵ which applies to monadic second-order variables to produce complex singular terms $(\epsilon(X))$. The theory involves, as its logical part, the axioms and inference rules of non-inclusive negative free logic with identity (NF$^=$):

NF1) $\forall v\alpha \rightarrow (E!t \rightarrow \alpha(t/v))$;

NF2) $\exists v E!v$;

NF3) $s = t \rightarrow (\alpha \rightarrow \alpha(t//s))$;

NF4) $\forall v(v = v)$;

NF5) $P\tau_1, ..., \tau_n \rightarrow E!\tau_i$ (with $1 \leq i \leq n$);

∀I): $E!a...\phi(a/x) \vdash \forall x\phi$;

∀E): $\forall x\phi, E!a \vdash \phi(a/x)$;

∃I): $\phi(a/x), E!a \vdash \exists x\phi$;

⋆We can also define the other connectives and the universal $\forall xAx =_{def} \neg\exists x\neg Ax$.

∃E): $\phi(a/x), E!a...\psi, \exists x\phi \vdash \psi$, where a is a new individual constant which does not occur in ϕ and ψ.

Additionally, the theory involves an axiom-schema of universal instantiation for second-order variables ($\forall X\phi(X) \rightarrow \phi(Y)$), a rule of universal generalisation (GEN), a second-order comprehension axiom schema (CA: $\exists X\forall x(Xx \leftrightarrow \alpha)$) and *modus ponens* (MP)*.

The abstraction principle that characterizes this theory is obtained by weakening BLVa (arbitrarily interpreted) by means of the condition of Permutation Invariance (cf. [1], [3]).

W-BLV: $\forall F\forall G(\epsilon F = \epsilon G \leftrightarrow \forall x(Fx \leftrightarrow Gx) \wedge \epsilon(\pi(F)) = \pi(\epsilon F))$

As well known, ϵ operator (as defined by standard BLV), also arbitrarily interpreted, is not Permutation Invariant – because, roughly speaking, by being inconsistent is unable to define or rule any function. We can emphasize that, given an arbitrary interpretation, Permutation Invariance fails precisely for the argument that determines its inconsistency. In other words, as can be pointed out for other consistent revisions of BLV, in any case in which it is safely restricted, ϵ turns out to satisfy Permutation Invariance, namely it is such that $\pi(\epsilon) = \epsilon$, i.e. $\forall X\forall y(\epsilon X = y \leftrightarrow \epsilon(\pi(X)) = \pi(y))$. Then, the second conjunct of the right-hand side of W-BLV requires that – no matter which object y is identical to ϵF – ϵ satisfies Permutation Invariance for the considered arguments*.

Accordingly, W-BLV, as a bi-conditional, turns out to be satisfied by any concept instantiating the universal quantifier. On the one side, given an arbitrary interpretation of the abstraction operator, for any concept different from Russellian concept (R), $\pi(\epsilon) = \epsilon$. On the other side, we can consider Russell's Paradox as a *reductio ad absurdum* of the alleged truth of both the side of the bi-conditional for the concept R: the contradiction proves that ϵR – as legitimately admitted on a free logical background – does not exists, namely it is a term devoid of denotation; accordingly, it is not identical to itself (so, falsifying the left-hand side of W-BLV) and, even if R, as any other concept, is co-extensional with itself, it falsifies Permutation Invariance of the operator*. Accordingly, also the right-hand side of W-BLV is false and also the instance of the bi-conditional for the concept R is verified.

Such a restricted version of W-BLV allow us to derive a corresponding restricted version of Hume's Principle. Nevertheless, the same restriction, on HP, is trivially satisfied by any instantiation, so it actually does not represent a weakening of the principle itself and allow us to derive the main arithmetical results, including Frege's Theorem.

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*From these axioms we can also derive the following theorems: T1) $\forall xE!x$; T2) $t = t \leftrightarrow E!t$; T3) $(\neg E!s \wedge \neg E!t) \rightarrow (\alpha \rightarrow \alpha(t//s))$.

*This revision of BLV (particularly of BLVa) is featured by a restriction that, respect to many other (syntactical ones), is expressible into the language. Indeed, the permutation π of the operator or of the concepts mentioned in the right-hand side of the bi-conditional can be defined as abbreviations of the effects of any first-order bi-jective function f: $D_1 \rightarrow D_1$ on the entities (sets, relations or functions) further up in the type hierarchy.

*This last claim follows from the definition of π and the result of non-existence of ϵR: on the one side, $\epsilon(\pi(R)) = \epsilon(\{\pi(x)|x \in R\}) = \epsilon(X)$ – where X is any other concept (based on π); on the other side, $\pi(\epsilon R)$, given that ϵR is not denoting, is another well-formed term without denotation; then, the identity between ϵX (for any X that is obtained by means of a permutation of R) and the empty term $\pi(\epsilon R)$ is false.

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Model theory of residue rings of models of Peano Arithmetic

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SPEAKER: Ludovica Conti

Let $\mathcal{M}$ be a model of Peano Arithmetic and $n \in \mathcal{M}$. We analyze the quotient rings $\mathcal{M}/n\mathcal{M}$ from a model theoretic point of view, see [2]. First of all we give a complete description in the case of n a prime power. Using the classical result due to Feferman-Vaught [3] connecting definability in products of first order structures to Boolean algebras we obtain information on the residue ring $\mathcal{M}/n\mathcal{M}$ for n composite, via a formalized version of the Chinese Remainder Theorem. A by product of our analysis of Feferman-Vaught theorem is a characterization of those unital commutative rings which are elementary equivalent to a "non trivial" product of commutative unital rings see [1]. This is a joint work with A. Macintyre.

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Barr exactness in classes of locally finite, transitive and reflexive Kripke frames

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SPEAKER: Matteo De Berardinis

Kripke frames (sets equipped with a binary relation) are one of the most popular semantics of modal logics (see [4] for a complete overview). They form the category **KFr**, where the arrows are the so called *p-morphisms*. Images via p-morphisms are called *p-morphic images* and such images are *generated subframes* of their codomains. A Kripke frame $\mathcal{F}$ is called *locally finite* if, for each $p \in \mathcal{F}$, the smallest generated subframe containing p is finite (in literature, *image finite* Kripke frames are better known; locally finite Kripke frames are those Kripke frames whose transitive closure is image finite). We are interested in $\mathbf{KFr}_{lf}$, the full subcategory of *locally finite* Kripke frames: this subcategory is closed under coproducts (disjoint unions), generated subframes and p-morphic images.

More generally, we are interested in any full subcategory $\mathcal{C} \subseteq \mathbf{KFr}_{lf}$ closed under the same operations (all colimits in $\mathcal{C}$ can be built from such operations). In [2], it has been shown that $\mathcal{C}$ is always *comonadic* over **Set**.

The algebraic semantics of modal logic is given by *modal algebras*. In the so called Thomason duality [3], $\mathbf{KFr}_{lf}$ corresponds to Pro$\mathbf{MA}_f$, the category of profinite modal algebras, with suitable morphisms, which is monadic over **Set** [2] (while image finite Kripke frames are dual to the topological modal algebras whose underlying topology is a Stone topology). Topological algebras and profiniteness are strictly related to classical problems such as canonical extensions of lattice-based algebras (among them are modal algebras). More generally, for any variety **V** of modal algebras generated by its finite members $\mathbf{V}_f$, the pro-completion [6] $\mathrm{Pro}\mathbf{V}_f$ is monadic over **Set**. In the above duality, $\mathrm{Pro}\mathbf{V}_f$ corresponds to the class of locally finite Kripke frames validating the equations defining **V**; the latter class has the aforementioned closure properties.

Our aim is to study categorical properties of classes of locally finite Kripke frames dual to $\mathrm{Pro}\mathbf{V}_f$, for some **V**. In particular, we want to characterize regularity and Barr exactness, at least under the assumption that the Kripke frames are transitive. Indeed, it is possible to prove that: (i) such classes have all limits (being the ind-completion of the class of finite Kripke frames belonging to it [2]) and (ii) under the assumption of transitivity, the usual image factorization gives an (extremal epi, mono)-factorization. Therefore, to establish regularity, it only remains to check that extremal epimorphisms are stable under pullbacks. We present a partial solution for the reflexive and transitive case.

From now on, we fix a full subcategory $\mathcal{C}$ of *reflexive and transitive* locally finite Kripke frames closed under disjoint unions, generated subframes and p-morphic images. In this case, the stability of extremal epimorphisms under pullbacks can be rephrased in terms of the dual of the *amalgamation property*. A *co-amalgamation* for a finite family $f_1, \ldots, f_n$ of epimorphisms with common codomain is a family $g_1, \ldots, g_n$ of epimorphisms with common domain, such that all the compositions $f_i g_i$ exist and coincide. The category $\mathcal{C}$ is said to satisfy the *co-amalgamation property* if each finite family of epimorphisms with common codomain has a co-amalgamation.

Co-amalgamation can be used to find out necessary conditions for regularity (following the classification in [5, Section 6.3], see also [8, 7]): if $\mathcal{C}$ is regular, then it is forced to contain Kripke frames that can be built using co-amalgamation and p-morphic images.

The construction of a binary product in $\mathcal{C}$ can be performed by induction following the universal model construction, well known in the modal logic literature — see [1]. This implies that the product of a pair of objects in $\mathcal{C}'$ is a generated subframe of the product computed in any $\mathcal{C}$ containing $\mathcal{C}'$. The two products might coincide, for example, when $\mathcal{C}' = \mathcal{C} \cap \mathbf{Pos}_{lf}$, where $\mathbf{Pos}_{lf}$ is the class of locally finite posets. If this is the case, $\mathcal{C}'$ is closed under pullbacks in $\mathcal{C}$, being always closed under equalizers. This observation allows us to conclude that, if $\mathcal{C}$ is regular, then all its subclasses closed under finite products in $\mathcal{C}$ must be regular; in particular, $\mathcal{C} \cap \mathbf{Pos}_{lf}$ has to be regular, too. A case analysis, based on the co-amalgamation property, shows that exctly 8 subclasses of $\mathbf{Pos}_{lf}$ are regular. Therefore, the regular $\mathcal{C}$ must intersect $\mathbf{Pos}_{lf}$ in one of the 8 classes above; applying again the co-amalgamation property, we obtain 49 possible cases.

Barr exactness can also be studied. Similarly to what happens for regularity, given two regular $\mathcal{C}' \subseteq \mathcal{C}$, with $\mathcal{C}'$ closed under finite products in $\mathcal{C}$, if $\mathcal{C}$ is exact then $\mathcal{C}'$ is exact, too. In particular, $\mathcal{C} \cap \mathbf{Pos}_{lf}$ is exact if $\mathcal{C}$ is so. After having excluded a certain number of cases, we show that $\mathcal{C}$ is exact if it only contains the empty frame, or it is one of the following:

1. $\{\mathcal{F} \mid \mathrm{ht}(\mathcal{F}) \leq 1 \;\&\; \delta^e(\mathcal{F}) \leq 1\} \cong \mathbf{Set}$;
2. $\{\mathcal{F} \mid \mathrm{ht}(\mathcal{F}) \leq 1 \;\&\; \delta^e(\mathcal{F}) \leq 2\} \cong \mathbb{Z}_2^+\text{-}\mathbf{Set}$;
3. $\{\mathcal{F} \mid \mathrm{ht}(\mathcal{F}) \leq 2 \;\&\; \mathrm{wt}(\mathcal{F}) \leq 1 \;\&\; \delta^i(\mathcal{F}) \leq 1 \;\&\; \delta^e(\mathcal{F}) \leq 1\} \cong \mathbb{Z}_2^\times\text{-}\mathbf{Set}$;

Where ht and wt give bound for cardinality of chains, resp. antichains, and δ^e and δ^i give bound for cardinality of external, resp. internal clusters.

We are currently working on a full characterization of exactness in the reflexive and transitive case and on a generalization of this characterization without the reflexivity condition. In the latter context, exactness could be encountered in some non trivial cases. An example is given by the class $\mathbf{GL}\text{-Lin}_{lf}$ of locally finite, transitive and irreflexive Kripke frames for which the restriction of the binary relation to each rooted generated subframe is a (irreflexive) linear order: $\mathbf{GL}\text{-Lin}_{lf}$ is indeed equivalent to the category of presheaves $\mathbf{Set}^{(\mathbb{N},\leq)^{\mathrm{op}}}$.

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An Application of Infinitary Universal Algebra to Universal Algebra

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SPEAKER: Matteo De Berardinis

This is part of a work done with A. Bucciarelli (Université Paris Cité), P.-L. Curien (CNRS, Université Paris Cité), and A. Salibra (Université Paris Cité).

Just as monoids serve to represent the composition of functions from a set A to itself, *clones* [4, Chapter 2] aim at modelling the composition of functions $A^n \to A$ of arbitrary finite arity $n \in \omega$. Clones hold significant importance in universal algebra, enabling the study of varieties in a way independent of their presentation (i.e. of their similarity type and equations). For instance, despite having different presentations, the varieties of Boolean algebras and Boolean rings are regarded as equivalent because they share the same clone of term-operations. Moreover, the categorical understanding of universal algebra is based upon the notion of *Lawvere theory* [7], which is nothing but a clone in disguise. In addition to their foundational significance, clones find applications in theoretical computer science. Specifically, a broad spectrum of decision problems, known as constraint satisfaction problems, can have their computational complexity entirely classified thanks to a clone associated to them [6, 2].

However, the study of clones presents challenges, primarily due to their ω-sorted structure, featuring a distinct sort for each finite arity. For a recent approach addressing this challenge and treating clones as one-sorted algebras, see [3]. Here we take seriously the suggestion put forth by Walter D. Neumann [8], and we study algebras, here labeled *ω-clones*, to model the composition of functions $A^\omega \to A$ of fixed arity ω. This approach enables us to gain the full expressive power of clones; however, it comes at the expense of departing from the realm of classical (finitary) universal algebra, permitting operations of arity ω. A ω-clone $\mathcal{C}$ is a set together with constants $\{\mathsf{e}_i : i \in \omega\}$ and an ω-ary operation q satisfying three axioms:

$$\begin{aligned} q(\mathsf{e}_i, x_0, \dots, x_n, \dots) &= x_i; \\ q(x, \mathsf{e}_0, \dots, \mathsf{e}_n, \dots) &= x; \\ q(q(x, y_0, y_1, \dots), z_0, z_1, \dots) &= q(x, q(y_0, z_0, z_1, \dots), \dots, q(y_n, z_0, z_1, \dots), \dots). \end{aligned}$$

Among ω-clones, a special role is played by the *functional ω-clone* $\mathcal{O}_A$ containing all the ω-ary functions over a fixed set A. Just as any monoid M is a transformation monoid of its underlying set via the action given by left multiplication, every ω-clone $\mathcal{C}$ is (isomorphic to) a functional ω-clone on its underlying set C [8].

Given a similarity type τ (of ω-ary operation symbols), we augment the type of ω-clones by introducing a constant f for each $f \in \tau$. Thus we form a connection between algebras of type τ and ω-clones *over* τ (i.e. where elements of τ are interpreted as constants). If $\mathbf{A}$ is an algebra of type τ, $\mathcal{O}_\mathbf{A}$ is the functional ω-clone on A with the set $\{f^\mathbf{A} : f \in \tau\}$ of constants (with *value domain* $\mathbf{A}$). Firstly, we show that the set $T_\tau(\omega)$ of terms over a countable set of generators can be endowed with the structure of a ω-clone, denoted by $\mathcal{N}_\tau$, which holds a role as the initial object in the category of ω-clones.

Then we describe a way to connect τ-algebras and ω-clones over τ, both individually and collectively. Individually, if $\mathbf{A}$ is a τ-algebra and $\mathcal{C}$ an ω-clone over τ, we define

- $\mathbf{A}^{\uparrow}$ as the image of the unique mapping $\mathcal{N}_\tau \to \mathcal{O}_{\mathbf{A}}$;
- $\mathcal{C}^{\downarrow}$ as the τ-algebra serving as the value domain of the representation $\mathcal{C} \hookrightarrow \mathcal{O}_{\mathcal{C}^{\downarrow}}$.

Collectively, if K is a class of τ-algebras and H a class of ω-clones, we define

- $\mathsf{K}^{\vartriangle}$ as the class of the functional ω-clones with value domain an element of K;
- $\mathsf{H}^{\triangledown}$ as the class of τ-algebras which appear as value domain of some element of H.

We prove that in both cases, starting from a variety, the resulting class is again a variety. We thus establish a correspondence between familiar concepts of universal algebra (on the left) and their translation into clone-algebraic notions (on the right); for instance:

the absolutely free algebra $\mathbf{T}_\tau(\omega)$	the ω-clone $\mathcal{N}_\tau$
the clone $Clo(\mathbf{A})$ of $\mathbf{A}$	the image of $\mathcal{N}_\tau \to \mathcal{O}_{\mathbf{A}}$
the theory $Th(\mathbf{A})$ of $\mathbf{A}$	the kernel of $\mathcal{N}_\tau \to \mathcal{O}_{\mathbf{A}}$.

This dictionary makes possible the recovery of some noteworthy results in universal algebra. Among these is the characterization of classes of algebras definable by sets of equations; we show that for a class K of algebras are equivalent:

1. K is variety;
2. K is an equational class;
3. $\mathsf{K} = \mathsf{K}^{\vartriangle\triangledown}$ and $\mathsf{K}^{\vartriangle}$ is a variety.

This constitutes an enrichment to Birkhoff's celebrated HSP Theorem [1]. The ω-clone $\mathcal{O}_{\mathbf{A}}$ naturally possesses a topology. By equipping functional ω-clones with this topology, we obtain a short proof of a recent result regarding a topological version of Birkhoff's Theorem [5, 9].

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Model theory of valued fields via explicit model constructions

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SPEAKER: Anna De Mase

Explicit constructions of models of the theory of a valued field are useful tools for understanding its model theory. Since Kaplansky’s work ([5]), it has been a topic of interest to characterize value fields in terms of fields of power series. In particular, Kaplansky proved that, under certain assumptions, an equicharacteristic valued field is isomorphic to a power series field. For the mixed characteristic case, in [3] the author, assuming the Continuum Hypothesis, provides a characterization, in terms of power series, of pseudo-complete finitely ramified valued fields with a fixed residue field k and valued in a $\mathbb{Z}$-group G, using a Hahn-like construction with coefficients in a finite extension of the Cohen field $C(k)$ of k ([2]). In this construction, the elements of the field are “twisted” power series, i.e. powers series whose product is defined by having an extra factor, given by a power of an element in the field of coefficients with minimal positive valuation. This generalizes a result by Ax and Kochen in [1], who provide a characterization of pseudo-complete valued fields elementarily equivalent to the field of p-adic numbers $\mathbb{Q}_p$. In this talk, we describe this Hahn-like construction and the following characterization in the finitely ramified case. Moreover, we will see some applications and how these constructions are used to characterize model complete valued fields in the various settings, depending on the characteristics of the valued field and the residue field ([4]).

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On the complexity of some topological properties in highly computable graphs

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SPEAKER: Valentino Delle Rose.

The problem of deciding whether a graph has an *Eulerian* path, namely a path visiting each edge of the graph exactly once, has a very long history in mathematics, dating back to the famous problem of *the seven bridges of Könisberg*, solved by Euler in 1736.

Around 200 years later, Erdős, Grünwald and Vászsonyi extended Euler's result by characterizing those graphs which admit an infinite Eulerian path. This characterization strongly relies on the *number of ends* of a graph, namely, the maximum number of infinite connected components which can be obtained by removing a finite number of edges. Indeed, only graphs with one or two ends can have Eulerian paths.

From the point of view of computability theory, their result highlights an important difference between computable graphs and *highly computable* graphs, the latter notion corresponding to computable and locally finite graphs where, in addition, one can uniformly compute the degree of each vertex. In fact, Bean showed that, while there are computable, locally finite graphs which admit an Eulerian path but no computable one, every highly computable graph admitting Eulerian paths must have a computable one.

However, deciding whether a given graph admits an Eulerian path at all is a difficult problem: as Kuske and Lohray have shown, such problem is Π^0_2-complete even when restricting to connected, highly computable graphs. Interestingly, this turns out to be the same difficulty of simply counting the number of ends in a highly computable graph.

Motivated by these considerations, we have studied how the difficulties of these two problems precisely relate. We have found that counting the ends of the graph indeed represents the hardest task when deciding the existence of an Eulerian path. More precisely, we have shown that:

1. deciding existence of Eulerian paths is only (2-c.e.)-complete when restricting to highly computable graphs with one end,

2. the same problem realizes precisely the m-degrees of Δ^0_2 sets in the case of highly computable graphs with two ends.

To get these results we have conducted a detailed analysis, which we believe of independent interest, of the computational hardness of what we call the *separation problem*: to decide whether a finite set of edges separates a connected and highly computable graph into two or more infinite connected components. Two results here are particularly relevant. On the one hand, we show that any function which takes as input a highly computable graph and outputs a finite set of edges separating the graph must compute the halting problem. On the other hand, the separation problem turns out to be (non-uniformly) decidable for highly computable graphs with finitely many ends: in fact, from the number of ends of a graph and a single maximally separating set, we can compute the whole collection of separating sets for this graph.

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Injectivity of the coherent model for a fragment of connected $MELL$ proof-nets

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SPEAKER: Raffaele Di Donna.

Linear logic, introduced by Jean-Yves Girard in [5] in 1987, is a refinement of both classical and intuitionistic logic in which formulas are treated as resources: the structural rules of contraction and weakening are restricted to formulas that are marked with modal operators and the semantic equivalence induced by the models of linear logic is non-trivial.

In this context, we ask ourselves the question of finding a canonical object representing the proofs in the same class of semantic equivalence. Formalization through proof-nets removes the redundant information of sequent calculus that concerns the order of application of the rules and allows us to define the cut-elimination procedure through local manipulations of graphs rather than global transformations of proof trees. Consequently, it is an appropriate formalism to study the dynamics of normalisation and to prove fundamental properties of the system such as strong normalisation: the fact that, given a proof-net R, every chain of reductions starting from R ends in a cut-free proof-net (see [5] and [8]).

Following the point of view of Curry-Howard's correspondence between proofs and programs and relying on the existing literature about the relationship between the dynamics of PCF and the models providing a mathematical representation of this language ([2], [6], [7] and [9] to name a few), we study the connections between proof-nets and their semantic interpretations.

Therefore, we focus on three fundamental notions: the syntactic, semantic and observational equivalences. The first one is intrinsic, whereas the others depend on the model and on the notion of observation we choose, respectively. Generally speaking, these three notions of equivalence are increasingly coarse: all models identify syntactically equivalent proofs and all reasonable notions of observational equivalence include the semantic equivalence of all models. A model is *injective* when the induced semantic equivalence coincides with syntactic equivalence, *fully abstract* when it coincides with observational equivalence. In general, full abstraction fails when some points of the model are not interpretations of proofs. A classic example is Scott's continuous model, that is not fully abstract for PCF since the "parallel or" function is not PCF-definable, as proven in [9]. On the other hand, when every point of a model is the interpretation of a proof, we say that *full completeness* holds. This last property, which was originally studied in [1], is often exploited as a sufficient condition for full abstraction, for instance in [6].

In linear logic, the question of injectivity was addressed for the first time by Tortora de Falco in [10], where he produced two counter-examples to the injectivity of the multiset-based coherent model for multiplicative exponential linear logic without units ($MELL$). On the other hand, the injectivity of the relational model for the full multiplicative exponential fragment of linear logic was recently proven by de Carvalho in [4] by employing the powerful Taylor expansion technique, which allows us to represent a proof-net as the infinite series of its linear approximations.

To ask the question of injectivity is also a way to address the problem of proof identity: one asks whether two proofs are to be considered equal. In other words, one aims to specify what is a proof. As already mentioned, proof-nets identify distinct sequent calculus proofs that are morally the same, because they only differ in the order in which some rules are applied. We could then say that proof-nets capture more faithfully the essence of a proof. With the idea of "measuring" the quality of the representation of proofs as proof-nets, we study the question of injectivity for the coherent model in order to understand if one could make "more identifications" than proof-nets.

Advances

We resume the work on the injectivity of multiset-based coherent semantics which started in [10]. It was conjectured that the result of injectivity can be extended to all connected proof-nets and it was given a sufficient condition to reach this conclusion: the existence of an injective experiment for all connected proof-nets only consisting of axioms, tensors, derelictions and contractions. It was also proven that this condition is satisfied if one assumes that every contraction is terminal.

Atomic pre-experiment and (C)-pairs.

We can define a partial labelling of pairs of arcs, called atomic pre-experiment, in such a way that any two premises of an atomic contraction are incoherent and any two premises of a non-atomic contraction are incoherent or undefined. The definition relies on the fact that we're dealing with *connected* proof-nets: two arcs of the same type are incoherent if and only if, for every switching graph, the unique path linking the nodes of which these arcs are the conclusions contains neither of them. The atomic pre-experiment is an injective experiment when all contractions are atomic.

Under the same assumption, we generalize the notion of (C)-pair introduced in [10]. A pair of conclusions of atomic why not nodes with the same address is a (C)-pair if there is a switching graph in which at least one of them belongs to the unique path connecting the conclusions of the proof-net over which they are located. Once again, this definition requires that the proof-net is *connected*. When a pair of conclusions (a, a') of the proof-net has a unique (C)-pair, the atomic pre-experiment automatically guarantees the incoherence of (a, a').

Injectivity for connected $(?\parr)LL_{pol}$ proof-nets.
We now consider the fragment $(?\parr)LL_{pol}$ of linear logic that is defined by the following grammar:

$$N, M ::= X \mid ?X \mid ?P \parr N \mid N \parr ?P \qquad P, Q ::= X^{\perp} \mid !X^{\perp} \mid !N \otimes P \mid P \otimes !N$$

In this very specific framework, it turns out that the question of injectivity has a positive answer because, even when a pair of conclusions does not have a unique (C)-pair, we can always find one on which we can harmlessly assign incoherence.

Theorem 1. *Multiset-based coherent semantics is injective for connected $(?\parr)LL_{pol}$ proof-nets.*

Given that connected $(?\parr)LL_{pol}$ proof-nets embed the simply typed λI-calculus, which is the simply typed λ-calculus without weakenings (see [3]), we also have a proof of the following result.

Corollary 2. *Multiset-based coherent semantics is injective for the simply typed λI-calculus.*

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Against BHK interpretation of intuitionist logic. Towards a truthful foundation of this logic

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SPEAKER: Antonio Drago.

It is well-known that the formalists did not recognize any alternative to Brouwer's intuitionist program and its developments. Rather, they saw with satisfaction Heyting's formalization of intuitionist logic (IL) as an inclusion of the intuitionist suggestions into their formalist foundations of Logic and Mathematics. However, Goedel's theorems stopped Hilbert's formalist program. After this failure of formalism, the suggestion of the former rebel to Hilbert's program, Hermann Weyl, was accepted: not conceiving any alternative to Hilbert's program, it had to be pursued again, yet as a general reductivist project of a sort quite common in the philosophy of science at the time (Zach 2023, sect. 3).

In 1945 Stephen C. Kleene suggested the BHK interpretation of intuitionist logic (Dummett 1977, pp. 222–234; Coquand 2013; Iemhoff 2019, sect. 3.1). As a consequence of this commonly shared BHK interpretation it is usual to bring together the three philosophical attitudes of Brouwer, Heyting and Kolmogorov on the foundations of IL into a common view. Actually, this ecumenical viewpoint eventually refers to the axiomatic foundation of IL suggested by (Heyting 1930).

However, the BHK-interpretation is not a formal definition because the notion of "construction" is not defined and therefore open to different interpretations. Moreover, Kurt Goedel charged this interpretation of impredicavity (Goedel 1933, p. 53); later no conclusive remedy was offered.

A century of not clear prosecution of the research on the foundations followed. Someone lamented that after this century the attempts to advance in this subject were unsuccessful (Martin-Loef 2008). In my opinion the presently blurred borders of the Hilbert's program constitute the cause of what Dov Gabbay states: "The notion of a proof system is not well defined in the literature. There are some recognized methodologies such as 'Gentzen formulations', 'tableaux', 'Hilbert style axiomatic systems', but these are not sharply defined". (Gabbay 2014, p. 45; in addition, see the criticism of Contu 2006 and the criticism to proof theory by Dicher and Paoli 2021). Indeed, by not suspecting an alternative to the axiomatic system, all these techniques present mere "creative extensions" of Hilbert's formalist attitude.

Yet, already Beth (1959, sect 1.2), van Heijenoort and Hintikka interpreted Goedel's theorems as the need of discovering, "under penalty of misleading the research on the foundations", an alternative theoretical organization to Hilbert's axiomatic one. A previous paper (Drago 2012) recognized an alternative model of theoretical organization in many scientific theories: e.g. Lazare Carnot's mechanics, Sadi Carnot's thermodynamics, Lobachevsky's non-Euclidean geometry, Einstein's first theory of quanta, and Dirac's (first edition of his textbook on) quantum mechanics. This theoretical organization is called a problem-based one (PO). Moreover, a more radical and direct criticism to BHK interpretation came from the result of a previous paper (Drago 2021) proving that in 1932 paper Kolmogorov not only wanted to detach its foundation of IL from Heyting's axiomatic, but applied almost entirely the PO model of the alternative model of theoretical organization. Hence, the common appraisal of Kolmogorov's formalization as a merely intuitive foundation (since it is not an axiomatic one) of IL is denied by the fact that actually Kolmogorov's paper suggested a formalization of IL, yet of a new kind.

The model of a PO is composed of eight logical steps:

i) No more than the common knowledge on the field at issue is presupposed.

ii) A problem which is unsolvable through usual tools is declared; e.g., in Lobachevsky's geometry: how many parallel lines exist.

iii) The theory is aimed at discovering a new scientific method capable to solve the given problem.

iv) The theory makes use of doubly negated propositions whose corresponding affirmative propositions are not equivalent in meanings (DNPs).

v) By composing together DNPs the theory argues through *ad absurdum* arguments (AAA); the conclusion of each AAA is again a DNP (i.e. an AAA is a "weak" *ad absurdum* proof); it may work as a premise for a next AAA, so to obtain a chain of AAAs.

vi) The conclusion of the final AAA is a universal predicate, ¬¬UP, which represents a possible hint for the resolution of the given problem and all related problems.

vii) At this point as a matter of fact the author translates the above predicate ¬¬UP into the corresponding affirmative predicate UP. Apparently the author thinks to have collected by his previous reasoning enough evidence to be justified in promoting his conclusion ¬¬UT to the corresponding affirmative proposition UP, although this change is not allowed by intuitionist logic, which previously he had adhered to.

viii) Exactly because now this proposition UP is an affirmative one, the author can test against reality it and all the derived propositions obtained from it by means of classical logic; hence, he can validate his entire theory or not (Drago 2012).

The substantial adhesion of Kolmogorov's paper to the PO model suggests that only Kolmogorov's foundation formalizes IL in an alternative way to Hilbert's method of axiomatization whereas Heyting's axiomatization of IL according to Hilbert's method of axiomatization actually represents a reductive compromise between the intuitionist program and the formalist program (On the other hand, Brouwer did not want build an alternative logical system to the classical one; de Stigt 1990, sect. 5.8. However, he later was in agreement with Heyting's axiomatic of IL as well as his previous axiomatization of projective geometry; van Dalen 1990).

Hence, at present we have two alternative philosophical and formal views of logic and they are mutually incompatible, being the failure or not of the doubly negation law a sharp borderline between them (Prawitz and Melmnaas 1968). It is time to disregard previous compromises for starting again from the dichotomy intuitionism/formalism.

It has to be noticed that Kolmogorov's foundation of IL is partially unsatisfactory because it relies on mathematical problems, instead of a logical subject. Therefore, let us try to build a rational foundation of IL as an exact PO.

First of all one has to state the basic problem of IL. The best candidates are the following ones.

1) "It is not true that equivalence is not identity". This problem was suggested by Troelstra (1990) when illustrating at best the foundations of intuitionism. It is remarkably that already in 18th century earlier Leibniz and Condillac put the same problem as the key problem of whole science. However, this is an advanced problem which can be tackled not before one formalizes equivalence and identity.

2) Already Aristotle stated the basic problem of the system of each kind of logic: "to produce a system of arguing which does not lead to a contradiction."(Aristotle)

3) "Since "Nothing is without a reason" (Leibniz), how to expressed this reason by means of a logical implication?" This problem may be considered the motivation of a natural deduction theory.

4) "What means a negation?" Notice that by applying its double negation law classical logic denies this problem: a negation is defined as a logical operation which, by adding one more negation leads to an affirmation. Hence, the previous problem is a specific one of IL. Among the above listed problem the more convenient one appears the fourth one.

Now let us remark that it is possible to produce an infinite chain of negations; this chain is surely meaningless. One has to stop it. The simpler way is to ask about the second negation of this chain: are two negations equivalent to the corresponding affirmation, or not? Here a crossroads occurs: 1) either the double negation is equivalent to the affirmation (and hence for the first time a logical implication appears ¬¬A→A). 2) Or it is not equivalent and the double negation law fails. In the latter situation which is typical of IL one has to characterize the single negation. After Brouwer it is customary to identify intuitionist negation with its implication of absurd. But Kolmogorov (1932, p. 332) correctly remarked that in IL the implication of absurd cannot be exempted to be proved. Hence, the question is whether this proof is considered as given a priori, or potentially given, or to be exhibited. Raatikainen's analysis of this question as it is debated in the current literature on the foundations of IL could not solve the problem (Raatikainen 2004, p. 143).

However, even in the most likelihood case we do not dispose of a proof of implication of absurd, we can state that "It is not true that one cannot prove the absurd". An application of PSR, which in the PO model constitutes the final step – i.e. the application of PSR to the final conclusion of the possible chain of AAAs - translates previous proposition into: "Negation implies absurd", i.e. "One can a priori prove that negation implies the absurd", i.e. the first of the three cases considered by Raatikainen. This proposition represents the common attitude about the intuitionist negation, i.e. to assume an a priori proof of the implication of the absurd. In other words, the PSR is here applied to not the conclusion of a PO theory of IL but a single constant. This move preconceives the theory. After this application of PSR the problematic attitude of OP is cancelled and the author argues in classical logic. Therefore after this step a formalist foundation of IL starts, although along different premises from that of classical logic. In this way the problematic attitude is replaced by a formalist attitude; which made self-confident by its inclusion of an apparently different viewpoint, wrongly exchanged the problematic aspect of IL with its intuitive aspect. Being the formalists convinced that any intuitive aspect of logic has to be either included in their kind of formalism or suppressed as a primitive aspect here we recognize the common attitude of last century formalists.

In this sense, the polemic Hilbert-Poincaré about the principle of mathematical induction is enlightening. Hilbert called Poincaré's version of it a merely intuitive principle (van Heijenoort 1967, p. 480-481), whereas actually this version is a DNP (Drago 1996), i.e. the only version which is the suitable one for a PO theory.

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The Logic of Partitions with the Application to Information Theory and Quantum Mechanics

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SPEAKER: David Ellerman.

The Boolean logic of subsets is usually only presented in the special case of propositional logic (where the universe set is $U = 1$). Subsets and partitions (or quotient sets) are category-theoretic duals so there is a dual logic of partitions. The join and meet operations on partitions were defined in the 19th century (Dedekind and Schröder), but the definition of the implication operation on partitions and two algorithms for defining all the Boolean operations on partition only came in the 21st century ([1], [4]). This development of the logic of partitions led to a series of applications.

Just as Boole's first application was the quantitative version of the logic of subsets in finite probability theory, so the first application of partition logic was the quantitative version as the logical information theory based on the notion of logical entropy ([2], [3]). The logical entropy of a partition is a probability measure interpreted as the probability in two random draws from U that one will draw a distinction of the partition (i.e., a pair of elements in different blocks of the partition). All the usual Venn diagram definitions for simple, joint, conditional, and mutual logical entropy follow from it being a measure. The better-known Shannon entropy is not a measure on a set but there is a non-linear monotonic transformation of those compound logical entropy formulas that yields all the corresponding formulas for Shannon entropy. Thus logical entropy provides a new logical foundation for information theory which includes the Shannon entropy as a specialized formula that is central to coding and communications theory. Logical entropy also generalizes to the quantum level where it is the probability that in two independent measurements of the same prepared quantum state, two different eigenvalues will be obtained.

The second application of partition logic is to the century-old problem of interpreting quantum mechanics (QM). When the set-based concepts of partition mathematics are linearized to Hilbert spaces, then one arrives at the mathematical formalism of QM–not the physics of QM which is obtained by quantization. Since partitions are the mathematical tool to describe distinctions and indistinctions or definiteness and indefiniteness, this shows that QM math is the mathematics of (objective) indefiniteness and definiteness. One can even see the lattice of partitions on a set as the bare bones or skeletal version (i.e., stripping away the scalars) of the pure, mixed, and classical states in QM. Moreover, the other basic QM concept is that of an observable operator where the direct-sum decomposition of its eigenspaces is just the linearized version of the inverse-image partition of a numerical attribute $f : U \to \mathbb{R}$ Thus both the quantum states and observables are derived from partitions and the basic notion of projective measurement, measuring a state by an observable, is represented back at the set level by the join of the two partitions. This partitional way of interpreting the quantum formalism is the "Objective Indefiniteness or Literal interpretation" of quantum mechanics ([5], [6]).

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Well-ordered forests as spectra of Heyting algebras

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SPEAKER: Damiano Fornarsiere.

The problem of describing the prime spectra of distributive lattices was raised in [2, 7] and it is equivalent to the classical problem of characterizing the posets isomorphic to the spectrum of a commutative ring with unity [3, 8, 10]. In [6] Esakia asked the same question for Heyting algebras. Both questions remain unsolved.

Characterize the poset of prime ideals of a distributive lattice with a unit and/or a zero.
Chen and Grätzer – [2, 7].

Can an arbitrary partially ordered set be the partially ordered set of prime ideals in a ring?
Kaplansky – [10].

It is tempting to replace lattices by Heyting algebras and suggest this as a new problem.
Esakia – [6].

In what follows we explain these problems, the connections between them, and we state our contribution.

A *Heyting algebra* is a structure $\boldsymbol{A} = \langle A; \vee, \wedge, 0, 1, \to\rangle$ where $\langle A; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and $\to$ is a binary relation on A such that, for every $\{x, y, z\} \subseteq A$, it holds $x \leqslant y \to z$ if and only if $x \wedge y \leqslant z$ ($\leqslant$ is the lattice order of $\boldsymbol{A}$). A filter F of a lattice is said to be *prime* if it is proper and, if $x \vee y \in F$, then $x \in F$ or $y \in F$. Equivalently, a filter F is prime when its complement is an ideal. Similarly, an ideal of a lattice is said to be prime if its complement is a filter. As such, when ordered by inclusion, the posets of prime filters and of prime ideals of a lattice are dually isomorphic. The poset of prime filters of a lattice is called its *prime spectrum*.

Where $\langle X; \leqslant\rangle$ is a poset and $Y \subseteq X$, we use the following shorthands:

$$\downarrow Y = \{x \in X \colon \text{there is } y \in Y \text{ s.t. } x \leqslant y\}, \quad \uparrow Y = \{x \in X \colon \text{there is } y \in Y \text{ s.t. } y \leqslant x\}.$$

A subset Y of X is called a *downset* (resp. *upset*) if $\downarrow Y = Y$ (resp. $\uparrow Y = Y$).

Definition 1. A tuple $\langle X; \leqslant, \tau\rangle$ is called a *Priestley space* when $\langle X; \leqslant\rangle$ is a poset, $\langle X; \tau\rangle$ is a compact topological space, $x \nleqslant y$ implies that there exists a clopen upset U such that $x \in U$ and $y \notin U$. If, moreover, $\downarrow V \in \tau$ for every $V \in \tau$, then $\langle X; \leqslant, \tau\rangle$ is called an *Esakia space*.

When equipped with suitable morphisms, Priestley spaces (resp. Esakia spaces) form a category dually equivalent to that of bounded distributive lattices (resp. Heyting algebras) and their homomorphisms [5, 6, 13]. In particular, every prime spectrum of a bounded distributive lattice (resp. Heyting algebra) can be endowed with a topology turning it into a Priestley (resp. Esakia) space. A poset with this property will be called *Priestley* (resp. *Esakia*) *representable*. Consequently, the following fact holds true:

Theorem 2. *A poset is isomorphic to the spectrum of a bounded distributive lattice (resp. Heyting algebra) if and only if it is Priestley (resp. Esakia) representable.*

Since every Heyting algebra is a bounded distributive lattice, every Esakia representable poset is Priestley representable too. However, the converse does not hold: for instance, an important difference between the classes of Priestley and Esakia representable posets is that the former is closed under order duals, while the latter is not.

The spectrum of a commutative ring with unit is the poset of its prime ideals ordered by inclusion. It turns out that the spectra of commutative rings with unit and those of bounded distributive lattices are the same, up to isomorphism [14]. Actually, a stronger result holds: distributive lattices can be represented in terms of spectral spaces [16] (topological spaces homeomorphic to the set of prime ideals of a commutative ring with unit endowed with the Zariski topology [8]). As the categories of Priestley spaces and that of spectral spaces are isomorphic [3], Kaplansky's question is equivalent to that of Chen and Grätzer.

We aim to contribute to the representability problem as follows: two *necessary* conditions for a poset $\langle X; \leqslant\rangle$ to be Priestley representable are from [2, 7, 10]:

1. Every nonempty chain of $\langle X; \leqslant\rangle$ has a supremum and an infimum in X;
2. For every $x, y \in X$, if $x < y$ there are $x_0, y_0 \in X$ such that $x \leqslant x_0 < y_0 \leqslant y$, but there is no $z \in X$ such that $x_0 < z < y_0$.

Notably, conditions (1) and (2) are not sufficient to ensure that a poset is Priestley representable, as proved in [8]. We will extend a third condition which first appeared in [8] and call it *order compactness*. We show that if a poset is order compact then it satisfies condition (1), but the converse is not true. Although a Priestley representable poset needs to be order compact and satisfy condition (2), these two conditions together do not imply that a poset is Priestley representable [12].

We remark that "abstract" characterizations of Priestley (not necessarily Esakia) representable posets exist: notably, the class of Priestley representable posets coincides with the class of profinite posets [9, 15]. However, understanding whether a poset is profinite seems as hard as checking if it is Priestley representable (for another abstract characterization, see [4]). For this reason, part of the interest shifted towards looking at certain subclasses of posets whose description is more transparent. For instance, Lewis showed that a *tree* (a poset with minimum and whose principal downsets are chains) is Priestley representable if and only if it satisfies conditions (1) and (2) [11].

On this spirit, we consider two classes of posets: (well-ordered) *forests*, that is, posets whose principal downsets are (well-ordered) chains, and *root systems*, the order duals of forests. We will prove that a root system is Priestley (resp. Esakia) representable if and only it satisfies conditions (1) and (2). As Priestley representable posets are closed under order duals, the above result yields a new proof of Lewis' description of Priestley representable forests [11]. This representation of root systems can also be used to simplify the proof of a result from [1].

However, as Esakia representable posets are not closed under order duals, a taxonomy of the Esakia representable forests remains open. Our main result is a characterization of the Esakia representable forests whose principal downsets are well-ordered:

Theorem 3. *A well-ordered forest is Esakia representable iff every nonempty chain has a supremum.*

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Computability of the Whitney Extension Theorems

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SPEAKER: Guido Gherardi.

Joint work with Andrea Brun and Alberto Marcone.

A very important topic in computable analysis, which, not surprisingly, has played a crucial role in the development of the subject, is the *effective* reformulation of classical mathematical theorems of the form

$$(\forall x \in X)(\exists y \in Y)A(x, y) \quad (*)$$

for X and Y suitable topological spaces. This approach exactly corresponds to showing that a corresponding multi-valued function, assigning to any $x \in X$ a $y \in Y$ such that $A(x, y)$, is *computable*.

A strictly related interesting aspect, also from a philosophical perspective, is to evaluate to what extent the classical proofs of those classical results have an algorithmic nature. Some of such proofs present indeed an intuitive computational flavour, but, so to say, some computable steps are hidden under the surface. Hence, in such cases, the proofs of computability of the corresponding multi-valued functions can retrace the classical proofs by showing in a rigorous way the computational content that was only sketched in them. Nevertheless, removing the rind to get to the computational pith might contain non-trivial steps, and it depends first of all on the choice of suitable translations of the classical concepts into computational notions.

An interesting example is given by the well-known Tietze-Urysohn Extension Theorem. A computable version of this theorem was proved by Klaus Weihrauch in [4]. In order to prove his result, Weihrauch provided in fact an *effectivization* of the classical proof contained in [1].

The Tietze-Urysohn Theorem classically finds a generalization in the Whitney Extension Theorem. For the real case, this theorem states that for any given (non-empty) closed set $F \subseteq \mathbf{R}^n$ and a *jet* of order m of functions on F, there exists a total continuous function g in $C_m(\mathbf{R}^n)$ such that both g as well as its partial derivatives coincide on F with the corresponding partial functions of the jet. Here a jet is a finite sequence of continuous functions defined on F satisfying Taylor's condition, and which behave like partial derivatives of each other. A classical proof of this statement is contained in [3] and, as a preliminary result, Stein proves an extension theorem for the limit case in which the jet consists only of a single continuous function, providing then another proof for the Tietze-Urysohn Extension Theorem.

In this talk I will check the computability of the construction of Whitney's extensions through an effectivization of the proofs given by Stein. A preliminary investigation of their computational content brought already in [2] to the systematic classification of different formulations of the projection point problem onto closed subsets of $\mathbf{R}^n$, and it turned out that only the problem of finding approximated projections of an $x \in \mathbf{R}^n$ onto a closed $F \subseteq \mathbf{R}^n$ up to a given error bound ε is computable with respect to full information for closed sets (which consists of an exact open covering of the complement of the set and of a dense subset of the set itself). In fact, the effectivization of Stein's proofs requires the use of approximations in several different aspects and in a way that implies a non-trivial departure from their original formulations.

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Logic for Artificial Intelligence: a teaching experience

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SPEAKER: Silvio Ghilardi.

Teaching Mathematical Logic within a curriculum oriented to Artificial Intelligence (AI) requires deep restructuring, concerning both the scientific content of the course and the lecture delivering mode. Given that logic is the essential part of the symbolic component of AI, it is clear that in an AI-specific logic course, high priority should be granted to computational tools that can be used in applications such as constraint solving, planning, knowledge representation, verification, multi-agent systems, etc.

The experience that this talk reports is located in a three-year inter-athenaeum AI course opened in the academic year 2021-2022 jointly by the Universities of Milan (Statale and Bicocca) and Pavia. Lectures are given in English and a significant part of students come from foreign countries (mostly East European and Middle East countries). Contributions to this initiative are currently given by mathematicians as well as by physicists and computer scientists. The Computational Logic course is located in the first semester of the first year of the study curiculum, where it constitutes the only teaching activity with mathematical content outside the analytic area.

This context made the structuring of the course rather challenging and peculiar. The main focus was on the SMT area ('Satisfiability Modulo Theories', see the website

`https://smtlib.cs.uiowa.edu/`

for information) and the related algorithms. These algorithms include the Davis-Putnam-Longemann-Loveland (DPLL) procedure (with relevant heuristics such as backjumping and conflict driven learning), as well as some decision procedures for quantifier-free (arithmetic and non-arithmetic) fragments, together with their combinations. Methods for handling quantifiers, through Herbrand instantiation or quantifier elimination, are also partially covered. From the mathematical logic point of view, Tarski's semantics for first-order logic is the fundamental formal reference framework.

The lab activities (2 hours per week out of 5 hours in total assigned to the course) formed an an essential part of the course and also of the exam tests. The z3 solver from Microsoft Research was adopted (the solver is freely available online for the different platforms). Students were guided in modeling and solving via computer problems of various nature, like puzzles, solitaire games (sudoku, domino, tetravex, peg), graphs problems (coloring, identification of Hamiltonian paths/cycles), scheduling and planning problems, verification of code fragments, etc.

The interdisciplinary aspect of the course was particularly emphasized: thanks to the extreme flexibility inherent to SMT tools, connections to disciplines like discrete mathematics, operations research, algorithms and data structures, programming languages were illustrated. Finally, the formalization of concrete examples turned out to be a very effective tool for our first-year students in order to access the basic mathematical language and the basic set-theoretic notions, whose knolwledge is essential for the solution of the proposed problems.

During the presentation of the talk, we shall supply a brief overview of the basic SMT notions as well as concrete examples of the problems carried out during the lab activities.

Interpolation failures in semilinear substructural logics

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SPEAKER: Valeria Giustarini.

In this contribution we present some new results concerning the deductive interpolation property in substructural logics whose equivalent algebraic semantics are classes of residuated lattices. Residuated structures play an important role in the field of algebraic logic; their equivalent algebraic semantics, in the sense of Blok and Pigozzi [1], encompass many of the interesting nonclassical logics: intuitionistic logic, intermediate logics, many-valued logics, relevance logics, linear logics and also classical logic as a limit case. Thus, the algebraic investigation of residuated lattices is a powerful tool in the systematic and comparative study of such logics.

Let us be more precise; a residuated lattice is an algebra $\mathbf{A} = (A, \vee, \wedge, \cdot, \backslash, /, 1)$ of type $(2, 2, 2, 2, 2, 0)$ such that: $(A, \vee, \wedge)$ is a lattice; $(A, \cdot, 1)$ is a monoid; the residuation law holds: for all $x, y, z \in A$, $x \cdot y \leq z \Leftrightarrow y \leq x\backslash z \Leftrightarrow x \leq z/y$, (where $\leq$ is the lattice ordering). Residuated lattices form a variety. A residuated lattice is said to be: *integral* if the monoidal identity is the top element of the lattice; *commutative* if the monoidal operation is commutative; n-potent if it holds that $x^n = x^{n+1}$; *semilinear* if it is a subdirect product of chains. Residuated lattices with an extra constant 0 are called FL-algebras (since they are the equivalent algebraic semantics of the Full Lambek calculus, see [3]), and we call them 0-*bounded* if $0 \leq x$; *bounded* if they are integral and 0-bounded. Semilinear bounded commutative FL-algebras are called MTL-algebras since they are the equivalent algebraic semantics of the monoidal t-norm based logic MTL [2].

Our results use one of the most interesting *bridge theorems* that are a consequence of algebraizability: the connection between logical interpolation properties and algebraic amalgamation properties. We say that a logic $\mathcal{L}$, associated to a consequence relation $\vdash$, has the *deductive interpolation property* if for any set of formulas $\Gamma \cup \{\psi\}$, if $\Gamma \vdash \psi$ then there exists a formula δ such that $\Gamma \vdash \delta$, $\delta \vdash \psi$ and the variables appearing in δ belong to the intersection of the variables appearing both in Γ and in ψ, in symbols $Var(\delta) \subseteq Var(\Gamma) \cap Var(\psi)$. If the logic $\mathcal{L}$ has a variety V has its equivalent algebraic semantics, and V satisfies the *congruence extension property* (CEP), $\mathcal{L}$ has the deductive interpolation property if and only if V has the *amalgamation property* (without the CEP, the amalgamation property corresponds to the stronger Robinson property, see [6]). Let us then recall the other necessary notions; given a class K of algebras in the same signature, a *V-formation* is a tuple $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ where $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathsf{K}$ and i, j are embeddings of $\mathbf{A}$ into $\mathbf{B}$ and $\mathbf{C}$ respectively; an *amalgam* in K for the V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ is a triple $(\mathbf{D}, h, k)$ where $\mathbf{D} \in \mathsf{K}$ and h and k are embeddings of respectively $\mathbf{B}$ and $\mathbf{C}$ into $\mathbf{D}$ such that $h \circ i = k \circ j$. A class K of algebras has the amalgamation property if for any V-formation in K there is an amalgam in K. We focus on the study of the amalgamation property in semilinear varieties of residuated lattices, solving some long-standing open problems; most importantly, we establish that semilinear commutative (integral) residuated lattices and their 0-bounded versions do not have the amalgamation property.

In order to obtain a failure of the amalgamation property, we use the recent results in [4]; the authors show that in a variety V with the CEP and whose class of finitely subdirectly irreducible members $\mathsf{V}_{\mathrm{FSI}}$ is closed under subalgebras, the amalgamation property of the variety is equivalent to the so-called *one-sided amalgamation property* of $\mathsf{V}_{\mathrm{FSI}}$. Given a V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$, a *one-sided amalgam* for it is a triple $(\mathbf{D}, h, k)$ with $\mathbf{D} \in \mathsf{K}$ and as for amalgamation $h \circ i = k \circ j$, but while h is an embedding, k is a homomorphism. A class K of algebras has the one-sided amalgamation property if for any V-formation there is a one-sided amalgam in K. The mentioned result of [4] is particularly useful in varieties generated by commutative residuated chains; indeed, all commutative residuated lattices have the CEP and a semilinear residuated lattice is finitely subdirectly irreducible if and only if it is totally ordered. Hence, in order to show the failure of the amalgamation property in a semilinear variety with the congruence extension property, it suffices to find a V-formation whose algebras are totally ordered, and that does not have a one-sided amalgam in residuated chains. We do exactly this, and we exhibit a V-formation, which we call $\mathcal{VS}$-formation, given by 2-potent commutative integral residuated chains that does not have a one-sided amalgam in the class of totally ordered residuated lattices. This entails that, if V is a variety of semilinear residuated lattices with the congruence extension property, and such that the algebras in the $\mathcal{VS}$-formation belong to V, then V does not have the amalgamation property. In particular we get the following new results.

Theorem 1. *The following varieties do not have the amalgamation property:*

1. *Semilinear commutative residuated lattices;*

2. *Semilinear commutative integral residuated lattices;*

3. *Semilinear commutative* FL*-algebras;*

4. *MTL-algebras;*

5. n*-potent MTL-algebras for* $n \geq 2$.

Using some algebraic constructions (*rotations* and *liftings*) we are also able to adapt our counterexample to construct a V-formation consisting of, respectively, involutive ($\neg\neg x = x$, where $\neg x = x\backslash 0$) and pseudocomplemented ($x \wedge \neg x = 0$) FL-algebras; thus in particular we obtain that involutive and pseudocomplemented MTL-algebras also do not have the amalgamation property.

Finally, we mention that the algebras involved in the V-formation that yields the counterexample can be constructed by means of a new construction that we introduce in order to be able to construct new chains from known ones. Such construction extends and generalizes the *partial gluing construction* introduced in [5], and allows us to find other countably many varieties of residuated lattices without the amalgamation property.

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Generalised Laver trees and Cohen κ-reals

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SPEAKER: Giorgio Laguzzi.

In this talk we focus on the relation between Laver generic κ-reals and Cohen κ-reals in the generalised Baire spaces. We prove that, in contrast with the standard Laver forcing $\mathbb{L}$ in ω^ω, any suitable generalisation $\mathbb{L}_\kappa$ in κ^κ adds Cohen κ-reals. We also study a dichotomy and an ideal naturally related to generalized Laver forcing. Using this dichotomy, we prove the following stronger result: if $\kappa^{<\kappa} = \kappa$, then every $< \kappa$-distributive tree forcing on κ^κ adding a dominating κ-real which is the image of the generic under a continuous function in the ground model, adds a Cohen κ-real. As a consequence of these constructions we also show that the generalised Laver measurability and the generalised Ramsey property imply the Baire property in the generalised Baire space for topologically reasonable families of subsets of κ^κ.

This is a joint work with Yurii Khomskii, Marlene Koelbing and Wolfgang Wohosfki.

Factorization in generalized power series

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SPEAKER: Noa Lavi.

A classical tool in the study of real closed fields are the fields $K((G))$ of generalised power series (i.e., formal sums with well-ordered support) with coefficients in a field K of characteristic 0 and exponents in an ordered abelian group G. We generalize previous results about irreducible elements and unique factorization in the subring $K((G^{\leq 0}))$.

The fibration of algebras

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SPEAKER: Fosco Loregian.

We study fibrations arising from indexed categories of the following form: fix two categories $\mathcal{A}, \mathcal{X}$ and a functor $F : \mathcal{A} \times \mathcal{X} \to \mathcal{X}$, so that to each $F_A = F(A, -)$ one can associate a category of algebras $\mathsf{Alg}_{\mathcal{X}}(F_A)$ – or an Eilenberg-Moore, or Kleisli category if each F_A is a monad. If $\int F$ is the Grothendieck construction applied to F, we call the functor $\int F \to \mathcal{A}$ (whose typical fibre over A is the category $\mathsf{Alg}_{\mathcal{X}}(F_A)$) the **fibration of algebras** obtained from F. Examples of such constructions arise in disparate areas of mathematics (the theory of formal languages, categorical logic, algebraic geometry and topology, computer science), and are unified by the intuition that $\int F$ is a form of *semidirect product* $\mathcal{A} \ltimes_F \mathcal{X}$ of the category $\mathcal{A}$, acting on $\mathcal{X}$, via the functor $F\colon \mathcal{A} \times \mathcal{X} \to \mathcal{X}$. This allows to draw an unexpected connection between representation-theoretic techniques and intuitions, and type-theoretic ones.

Let $\mathcal{A}, \mathcal{X}$ be two categories and $F\colon \mathcal{A} \times \mathcal{X} \to \mathcal{X}$ be a functor. Consider the assignments

- $A \mapsto \mathsf{Alg}(F_A)$ where $F_A := F(A, -) : \mathcal{X} \to \mathcal{X}$ is the functor 'saturated' in the parameter A and $\mathsf{Alg}(F_A)$ is its category of endofunctor algebras.
- Assuming that each F_A is a monad, and that a morphism $u : A \to A'$ induces a monad homomorphism $F_u : F_{A'} \Rightarrow F_A$, $A \mapsto \mathsf{EM}(F_A)$ where the latter is the Eilenberg-Moore category of F_A.

These define contravariant pseudofunctors $\mathcal{A}^{\mathrm{op}} \to \mathsf{Cat}$ which, as such, determine fibrations over $\mathcal{A}$, that we call respectively the **endofunctor algebra fibration**, and the **Eilenberg-Moore fibration** of F; we denote the domain of the fibration so obtained as $\mathcal{A} \ltimes_F \mathcal{X}$ or simply $\mathcal{A} \ltimes \mathcal{X}$ whenever F (the 'representation' of $\mathcal{A}$ on $\mathcal{X}$) is clear from the context.

Similar constructions provide the coEilenberg-Moore, coKleisli, endofunctor algebra... fibrations and opfibrations for a suitable parametric endofunctor F.

Among many examples of this construction we find:

- the fibration of modules [7] introduced by Quillen to study the cohomology of rings; this has $\mathcal{A} = \mathsf{Mon}(\mathcal{X})$ and $\mathcal{X}$ a (symmetric) monoidal category; we consider the Eilenberg-Moore fibration of algebras for the parametric monad $A \otimes \_$.
- the simple fibration [5, 4] known to type theorists, when $\mathcal{X} = \mathcal{A}$ is a Cartesian category acting on itself as $A \mapsto A \times \_$; we consider the coKleisli fibration of the parametric comonad $A \times \_$.[⋆]

[⋆]This should be seen as a categorified analogue of the *regular representation* action of each monoid on itself.

- the fibration of points [**?** 2, 1], obtained as evaluation at the terminal object for $\mathcal{X}^{\leftrightarrows'} \to \mathcal{X}$ from the freestanding split-epi category; a certain type of Eilenberg-Moore fibration captures the notion of *protomodular* category.

- the action of the category $\mathcal{A}$ of groups on the category $\mathcal{X}$ of k-Lie algebras given by $G \mapsto k[G] \otimes \_$; if we consider the Eilenberg-Moore fibration we get the celebrated Cartier-Gabriel-Konstant decomposition [3, 5.10.2] of the category ccHopf_k of cocommutative Hopf k-algebras as semidirect product $\mathsf{Grp} \ltimes \mathsf{Lie}_k$.

The present talk is meant to introduce the following theorems:

Theorem 1. *There exists a representation P of* Grp *on itself such that the semidirect product construction for monoids and groups is a functor $(G, H) \mapsto G \ltimes H : \mathsf{Grp} \ltimes_P \mathsf{Grp} \to \mathsf{Grp}$, which is a left adjoint. In the case of groups, the right adjoint is simply $r_c : G \mapsto (G, G, c : G \cdot G \to G)$ where c is the conjugation action and $G \cdot G$ is the free product of $|G|$ copies of G.*

Theorem 2. *Let $\mathcal{X}, \mathcal{A}$ have a zero object; let $T : \mathcal{A} \times \mathcal{X} \to \mathcal{X}$ be a parametric monad; then there is a short exact sequence*

$$1 \longrightarrow \mathcal{X} \xrightarrow{\Phi} \mathcal{A} \ltimes_T \mathcal{X} \xrightarrow{p^T} \mathcal{A} \longrightarrow 1$$

of left adjoint functors.

More in detail, the functors are obtained in the following way:

- p^T is the fibration of algebras generated by T and its right adjoint is given by fibrewise choice of the zero object 0_A (regarded as terminal) in each fiber $A \ltimes \mathcal{X} =: (p^T)^{-1}A$;

- Φ chooses the free algebra $(T_\varnothing X, \mu^\varnothing_X)$ in the fiber over the zero object 0 (regarded as initial), and its right adjoint V is determined by the forgetful sending a T_A-algebra $(X, \xi^A : T_A X \to X)$ to X;

- $1 \to \mathcal{X}$ and $\mathcal{A} \to 1$ are left adjoints, given that both $\mathcal{X}, \mathcal{A}$ have a zero object.

This provides backup for the notation $\mathcal{A} \ltimes_T \mathcal{X}$, since one can really imagine the category $\mathcal{A} \ltimes \mathcal{X}$ as obtained from the semidirect product of $\mathcal{A}$ acting on $\mathcal{X}$ via the representation T, or in other words as an *extension* of $\mathcal{A}$ by $\mathcal{X}$ given by T.

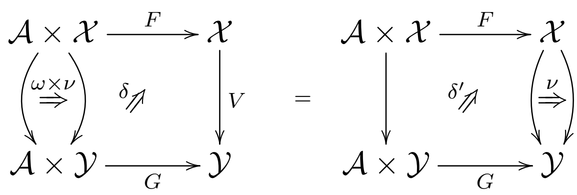
Moreover, $\mathcal{A} \ltimes_T \mathcal{X}$ is (2-)functorial on a (2-)category $\mathsf{Cat} \ltimes_\ell \mathsf{Cat} \to \mathsf{Cat}$ having

1. objects the triples $(\mathcal{A}, \mathcal{X}, F)$ where $F : \mathcal{A} \times \mathcal{X} \to \mathcal{X}$ is an endofunctor;

2. 1-cells are *oplax morphisms of algebras* i.e. pairs $(U, V, \delta) : (\mathcal{A}, \mathcal{X}, F) \to (\mathcal{B}, \mathcal{Y}, G)$ where $U : \mathcal{A} \to \mathcal{B}$, $V : \mathcal{X} \to \mathcal{Y}$ are functors and δ is a 2-cell filling the diagram below.

$$\begin{array}{ccc} \mathcal{A} \times \mathcal{X} & \xrightarrow{F} & \mathcal{X} \\ {\scriptstyle U \times V}\downarrow & \overset{\delta}{\nearrow} & \downarrow{\scriptstyle U} \\ \mathcal{B} \times \mathcal{Y} & \xrightarrow[G]{} & \mathcal{Y} \end{array}$$

3. 2-cells $(U, V, \delta) \Rightarrow (U', V', \delta')$ are pairs $\omega : U \Rightarrow U', \nu : V \Rightarrow V'$ of natural transformations such that the following equality of pasting 2-cells holds.



The category so defined arises as the fibration of lax algebras for the 'lax regular representation' $\mathsf{Cat} \times \mathsf{Cat} \to \mathsf{Cat}$ of the 2-category Cat on itself: in other words, to each endofunctor $\mathcal{A} \times \_$ of Cat we associate the category of oplax algebras having objects, 1-cells, 2-cells as in 1,2,3 above.

Theorem 3. *The correspondence $(\mathcal{A}, \mathcal{X}) \mapsto \mathcal{A} \ltimes \mathcal{X}$ is a 2-functor $\_ \ltimes \_ : \mathsf{Cat} \ltimes_\ell \mathsf{Cat} \to$* Cat.

Startin from Theorem 3 we can find a functor

$$\mathcal{A} \ltimes \mathcal{X} \xrightarrow{\langle p,V \rangle} \mathcal{A} \times \mathcal{X}$$

with a left adjoint L; then, invocation of Beck's monadicity theorem, or a straightforward direct check, show that $\mathcal{A} \ltimes \mathcal{X}$ is the Eilenberg-Moore object of the monad $\langle p, V \rangle \circ L$ generated by this adjunction, in the 2-category of fibrations over $\mathcal{A}$, over the object $(\mathcal{A} \times \mathcal{X}, \pi_\mathcal{A})$ (the 'trivial fibration' that projects (X, A) on A); indeed, the data of a monad like this amount exactly to a family $U_A : \mathcal{E}_A \to \mathcal{X}$ of monadic functors, realizing each of the $\mathcal{E}_A$ as the category of algebras of a monad $T_A : \mathcal{X}_{(=\pi^{-1}A)} \to \mathcal{X}_{(=\pi^{-1}A)}$.

An immediate consequence of this, is that limits in $\mathcal{A} \ltimes_T \mathcal{X}$ are created in the product $\mathcal{A} \times \mathcal{X}$, and thus are computed as follows:

- consider a diagram $D : \mathcal{J} \to \mathcal{A} \ltimes_T \mathcal{X}$, made of triples $(A_J; X_J, \xi_J)$;
- compute the limit $A := \lim A_J$ in $\mathcal{A}$, and reindex the A_J-algebra (X_J, ξ_J) into the fiber over A using the reindexings $u_J^* : \mathsf{EM}(T_{A_j}) \to \mathsf{EM}(T_A)$ induced by the terminal cone $u_J : A \to A_J$;
- compute the limit of $u_J^*(X_J, \xi_J)$ in $\mathsf{EM}(T_A)$ (i.e., in $\mathcal{X}$).

Extending this characterization, one notices that a parametric monad $T : \mathcal{A} \times \mathcal{X} \to \mathcal{X}$ is nothing but a 1-cell in the coKleisli 2-category of the 2-comonad $\mathcal{A} \times \_$; and that the presence of units $\eta_X^A : X \to T_A X$ and multiplication $\mu_X^A : T_A T_A X \to T_A X$ for every $(A, X) \in \mathcal{A} \times \mathcal{X}$ amounts to a pair of 2-cells in the same coKleisli 2-category.

Putting all together we have:

Theorem 4. *The following conditions are equivalent for a fibration $p : \mathcal{E} \to \mathcal{A}$:*

- *$\mathcal{E} \cong \mathcal{A} \ltimes \mathcal{X}$ for a parametric monad $T : \mathcal{A} \times \mathcal{X} \to \mathcal{X}$;*
- *p is monadic over the trivial fibration $\pi_\mathcal{A} : \mathcal{A} \times \mathcal{X} \to \mathcal{A}$.*

Hence, the following pieces of data are equivalent:

- *a parametric monad* $T : \mathcal{A} \times \mathcal{X} \to \mathcal{X}$*, i.e. (upon currying) a functor* $\mathcal{A} \to [\mathcal{X}, \mathcal{X}_\mu]$*;*
- *a monad* $T : (\mathcal{A} \times \mathcal{X}, \pi_\mathcal{A}) \to (\mathcal{A} \times \mathcal{X}, \pi_\mathcal{A})$ *in the 2-category* $\mathsf{Fib}(\mathcal{A})$*;*
- *a monad* $T : \mathcal{A} \times \mathcal{X} \to \mathcal{X}$ *in the 2-coKleisli category of the 2-comonad* $\mathcal{A} \times -$*, over the object* $\mathcal{X}$*;*
- *a monad in the (domain of the) 2-fibration* $\mathsf{Cat} \ltimes_\ell \mathsf{Cat}$ *of Theorem 3.*

Theorem 5. *Let* $\mathcal{X}, \mathcal{A}$ *be categories with an initial and a terminal object. Consider the category* $\mathsf{Fib}_{0,1}(\mathcal{A})$ *of fibrations over* $\mathcal{A}$ *admitting an initial and a terminal object, and the category of* reduced *parametric monads over* $\mathcal{A}$*, i.e. such that* $T(\varnothing, -)$ *is the identity functor. Then there is an adjunction*

$$\mathsf{Fib}_{0,1}(\mathcal{A}) \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathsf{RedMnd}(\mathcal{A})$$

whose unit η_p *is an equivalence iff* p *is the Eilenberg-Moore fibration of a parametric reduced monad* T*.*

Proof. The assumptions on a fibration $p : \mathcal{E} \to \mathcal{A}$ in $\mathsf{Fib}_{0,1}(\mathcal{A})$ are sufficient to write the following diagram:

$$\begin{array}{ccccc} \mathcal{E}_\varnothing & \underset{i_R}{\overset{i}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} & \mathcal{E} & \underset{p}{\overset{p_L}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} & \mathcal{A} \\ & & {\scriptstyle \langle p, i_R \rangle}\big\downarrow & & \\ & & \mathcal{E}_\varnothing \times \mathcal{A} & & \end{array}$$

from which we obtain a left adjoint for $\langle p, i_R \rangle$ as coproduct $i + p_L : (Y, A) \mapsto iY + p_L A$; then the monad of this adjunction is a (reduced, parametric) monad

$$\mathcal{E}_\varnothing \times \mathcal{A} \xrightarrow{i+p_L} \mathcal{E} \xrightarrow{\langle p, i_R \rangle} \mathcal{E}_\varnothing \times \mathcal{A}$$

where the parameters $\mathcal{A}$ act on the fiber $\mathcal{E}_0$ of p over the initial object 0.

Conversely, every reduced parametric monad $T : \mathcal{A} \times \mathcal{X} \to \mathcal{X}$ has a fibration of Eilenberg-Moore algebras, which is in fact an object of $\mathsf{Fib}_{0,1}(\mathcal{A})$. □

Note that:

- the correspondence so defined is a 2-adjunction, as it is defined (on 1- and) 2-cells;
- it is simply through a direct computation that one shows that the counit of this adjunction, sending a fibration of EM-algebras, to a parametric monad, and then to the fibration of EM-algebras of that monad, recovers the same fibration (the universal property of an EM-object plays an essential role).

This presents, and expands on, results of a work in progress with: **Danel Ahman** (University of Tartu, Estonia); **Davide Castelnovo** (Università degli studi di Padova); **Greta Coraglia** (Università degli Studi di Milano); **Nelson Martins-Ferreira** (Politécnico de Leiria, Portugal); **Ülo Reimaa** (University of Tartu, Estonia).

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Self-divisible ultrafilters and congruences in $\beta\mathbb{Z}$

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SPEAKER: Lorenzo Luperi.

We introduce *self-divisible* ultrafilters, which we prove to be precisely those w such that the weak congruence relation $\equiv_w$ introduced by Šobot is an equivalence relation on $\beta\mathbb{Z}$. We provide several examples and additional characterisations; notably we show that w is self-divisible if and only if $\equiv_w$ coincides with Šobot's strong congruence w, if and only if the quotient $(\beta\mathbb{Z}, \oplus)/w$ is a profinite group. We also construct an ultrafilter w such that $\equiv_w$ fails to be symmetric, and describe the interaction between the aforementioned quotient and the profinite completion $\hat{\mathbb{Z}}$ of the integers.

This is a joint work with Mauro Di Nasso, Rosario Mennuni, Moreno Pierobon and Mariaclara Ragosta.

Consequentia mirabilis as strong stability

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SPEAKER: Paolo Maffezioli.

The atomic formulas of a logic are decidable if they satisfy the law of excluded middle, namely $P \vee \neg P$ is a theorem. The atomic formulas of a logic are stable if the satisfy the law of double-negation elimination, namely $\neg\neg P \supset P$ is a theorem. While in classical logic decidability and stability coincide, in intuitionistic logic we may have stable atomic formulas that are not decidable. In [2] decidability is formulated a sequent-calculus rule corresponding to the law of excluded middle for atomic formulas:

$$\frac{P \Rightarrow C \quad \neg P \Rightarrow C}{\Gamma \Rightarrow C}\ dc$$

It is shown that when such a rule is added on top of the rules of the single-succedent calculus for propositional intutionistic logic Int, then the resulting calculus $\mathsf{Int} + dc$ is complete for classical propositional logic. Similarly, stability is formulated as a rule corresponding to the law of double-negation elimination for atomic formulas:

$$\frac{\neg P, \Gamma \Rightarrow \bot}{\Gamma \Rightarrow P}\ st$$

However, $\mathsf{Int} + st$ is *not* complete for classical propositional logic, since the sequent $\Rightarrow P \vee \neg P$ is not derivable in $\mathsf{Int} + st$. And since the rule st is admissible in $\mathsf{Int} + dc$, it follows that $Th(\mathsf{Int} + st) \subset Th(\mathsf{Int} + dc)$. At the same time, we also have $Th(\mathsf{Int}) \subset Th(\mathsf{Int} + st)$, since the sequent $\Rightarrow \neg\neg P \supset P$ is derivable in $\mathsf{Int} + st$ but clearly not in Int. This is why $\mathsf{Int} + st$ can be legitimately thought of as an intermediate logic, called stable logic in [2].

In this work I introduce a stronger form of stability, inspired by the law of classical logic $(\neg P \supset P) \supset P$ known as known as *consequentia mirabilis* [1]. By proceeding in a parallel fashion as in [2], we may consider extending Int with a rule corresponding to the *consequentia mirabilis* for atomic formulas:

$$\frac{\neg P, \Gamma \Rightarrow P}{\Gamma \Rightarrow P}\ cm$$

The calculus $\mathsf{Int} + cm$, too, is incomplete for classical propositional logic. Hence, $Th(\mathsf{Int} + cm) \subset Th(\mathsf{Int} + d)$. At the same time, $\mathsf{Int} + cm$ is deductively stronger than $\mathsf{Int} + st$, in the sense that $Th(\mathsf{Int} + st) \subset Th(\mathsf{Int} + cm)$. Indeed, the rule st is admissible in $\mathsf{Int} + cm$ but the sequent $\Rightarrow (\neg P \supset P) \supset P$ is not derivable in $\mathsf{Int} + st$. This motivates the study of a novel intermediate logic based on the *consequentia mirabilis* understood as a stronger form of stability. While I will focus mostly on the proof-theoretic properties of the calculus $\mathsf{Int} + cm$ (cut elimination, admissibility of the structural rules), I will also try to suggest, at least conceptually, that stability can be considered as weak form of strong stability just like the weak excluded middle is a weak form of of excluded middle.

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Bayesian Proof Nets: an encoding of Bayesian networks in to proof nets of linear logic

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SPEAKER: Roberto Maieli.

We propose an encoding of Bayesian network (BN) in proof nets (PN) of linear logic. Modularity of PNs allows to express efficiently graphical reasoning on probabilities as factorization of the joint probabilities of BNs.

Introduction

The role of graphs in probabilistic and statistical modeling [9] is multiple: they facilitate both the representations of joint probability functions and the efficient inferences (or reasoning) from observations. Graphical probabilistic models exploit both probability theory and graph theory. Graphs grasp the qualitative part of model: nodes represent events/random variables, edges represent dependencies between them, and conditional independence can be seen in graph. Probability grasps the quantitative part of model: local information about nodes and its neighbors, the strength of dependency, way of inference, ecc.. Similarly, the role of graphs in logical proofs modelling presents notable advantages: they simplify the syntactical representation of proofs and they facilitate well known proof-theoretical procedures as e.g. proof construction (or proof search) and proof reduction (or cut elimination).

Bayesian Networks [9, 2] are elegant and efficient graphical representations of probability distributions over random (discrete) variables. They are widely used in Artificial Intelligence and Machine Learning for different purposes such as automatic text classification, spam detection, sentiment analysis, supporting medical diagnosis, etc.. Bayesian Networks are very useful for inferring (or deducing) the probability of some events from other observed events (evidences). For this reason it appears natural to try to encode Bayesian Networks in logical systems in which it is possible to formalize reasoning on probabilities. Integration of Bayesian networks and first-order logic have been widely investigated, see e.g. [10], although it seems quite natural to express Bayesian Networks in classical propositional logic.

In this work we show a correspondence between (Boolean) Bayesian Networks and a special class of Proof Nets of the multiplicative-additive (MALL) fragment with Mix rule of propositional Linear Logic [3, 6], a constructive refinement of Classical Logic: this is called the class of Bayesian Proof Nets.

Proof Nets are a graphical syntax that allow the demonstrations of linear logic to be expressed in a modular and parallel way, abstracting away from the often useless bureaucracy of the sequent calculus. Actually, it is not a novelty to propose variants of proof nets inspired by the methodologies used in Machine Learning or more in general by Artificial Intelligence as e.g. the neural variant of proof nets based on Sinkhorn networks by [5].

The novelty of Bayesian Proof Nets is given by the fact they are special Proof Nets endowed with a probability distribution over the set of "switchings" (slices) associated to the Proof Nets. Switchings are kinds of interactive tests that play a double role: *(i)* they detect the "correctness" of any given PN and *(ii)* they infer the (joint) probability distribution associated to a PN from the (conditional) probability distribution associated to the links which the PN is built on. We know [9] that any BN $\mathbf{N}$ with n random (discrete) variables $A_1, ..., A_n$ expresses the joint probability $Pr(A_1, ..., A_n)$: here we show that $\mathbf{N}$ can be encoded by a probabilistic PN π^N built by n special Bayesian links: exactly one Bayesian link α_i encodes a random variable/node A_i of $\mathbf{N}$; then, each additive slice $S(\pi)$ (consisting of an appropriate mutilation of each B-link of π) corresponds to the factorization of the joint probability that is: $Pr(A_1, ..., A_n) = \prod_{i=1}^{n} Pr(A_i \mid \mathbf{Pa}(A_i)) = \prod_{i=1}^{n} \langle S(\alpha_i) \rangle$ where "$\langle S(\alpha_i) \rangle$" denotes the conditional probability associated to any switched Bayesian link $S(\alpha_i)$.

An extended and detailed version of this work can be found in [8] (accepted paper at DCAI2024 `https://www.dcai-conference.net/`).

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The Proof Theory of π-Calculus

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SPEAKER: Giulia Manara.

The π-calculus is a process calculus that models channel names communication between several participants [21]. It is considered to be the formal language of concurrent computation in the same way the λ-calculus is considered to be the one of functional programming. The grammar includes primitives for implement parallel composition, choice between actions and scope restriction, allowing the description of distributed systems. Despite its simple design, functional programming can be encoded in it as well as higher order process-passing calculi.

Processes	
$P, Q, R := \mathsf{Nil}$	nil
$\mid x!\langle y\rangle.P$	send (y over x)
$\mid x?(y).P$	receive (y over x)
$\mid P \mid Q$	parallel
$\mid (\nu x)P$	nu
$\mid x \triangleleft \{\ell : P_\ell\}_{\ell\in L}$	branching
$\mid x \triangleright \{\ell : P_\ell\}_{\ell\in L}$	selection

Com :	$x!\langle a\rangle.P \mid y?(b).Q \to P \mid Q\,[a/b]$	
Choice :	$x \triangleleft \{\ell : P_\ell\}_{\ell\in L_1} \mid x \triangleright \{\ell : Q_\ell\}_{\ell\in L_2} \to P_{\ell_k} \mid Q_{\ell_k}$	if $\ell_k \in L_1 \cap L_2$
Res :	$(\nu x)P \to (\nu x)P'$	if $P \to P'$
Par :	$P \mid Q \to P' \mid Q$	if $P \to P'$
Struc :	$P \to Q$	if $P \equiv P' \to Q' \equiv Q$

$P \mid Q \equiv Q \mid P$	$P \mid \mathsf{Nil} \equiv P$	
$(P \mid Q) \mid R \equiv P \mid (Q \mid R)$	$(\nu x)S \equiv S$	with $x \in \mathsf{free}(S)$
$(\nu x)(\nu y)P \equiv (\nu y)(\nu w)P$	$(\nu x)P \mid S \Rrightarrow (\nu x)(P \mid S)$	

Figure 1: Syntax and semantics of the π-calculus.

In this talk we discuss the language of the recursion-free fragment of the π-calculus (as presented in [6]) under the lens of proof theory. For this purpose, we introduce the sequent calculus PiL, an extension of Girard's *first order multiplicative and additive linear logic* MALL_1 [7] where sub-exponential modalities (as the ones studied in, e.g., [10, 11]), a non-commutative non-associative connective ($\blacktriangleleft$), and nominal quantifiers (И and Я) are introduced to model specific operational behaviors of the operators of the π-calculus.

	Formulas	
$A, B :=$	$\circ$	unit
$\mid$	a	atom
$\mid$	$\overline{a}$	negated atom
$\mid$	$A \parr B$	par
$\mid$	$A \otimes B$	tensor
$\mid$	$A \blacktriangleleft B$	prec
$\mid$	$A \oplus B$	oplus
$\mid$	$A \mathbin{\&} B$	with
$\mid$	$?_a A$	why-not
$\mid$	$!_a A$	of-course
$\mid$	$\forall x.A$	for all
$\mid$	$\exists x.A$	exists
$\mid$	$\text{И}x.A$	new
$\mid$	$\text{Я}x.A$	ya

$$\text{ax}\,\frac{}{\vdash a, \overline{a}} \qquad \parr\,\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B} \qquad \otimes\,\frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta} \qquad \text{cut}\,\frac{\vdash \Gamma, A \quad \vdash \overline{A}, \Delta}{\vdash \Gamma, \Delta}$$

$$\oplus\,\frac{\vdash \Gamma, A_i}{\vdash \Gamma, A_1 \oplus A_2}\, k \in \{1,2\} \qquad \&\,\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \mathbin{\&} B} \qquad \exists\,\frac{\vdash \Gamma, A\,[c/x]}{\vdash \Gamma, \exists x.A[x]} \qquad \forall\,\frac{\vdash \Gamma, A\,[y/x]}{\vdash \Gamma, \forall x.A[x]}\,\dagger$$

$$\circ\,\frac{}{\vdash \circ} \qquad \text{mix}\,\frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta} \qquad \blacktriangleleft^{\circ}\,\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \blacktriangleleft B} \qquad \blacktriangleleft\,\frac{\vdash \Gamma, A, C \quad \vdash \Delta, B, D}{\vdash \Gamma, \Delta, A \blacktriangleleft B, C \blacktriangleleft D}$$

$$\text{M}\,\frac{\vdash A, B}{\vdash ?_a A, !_a B} \qquad \text{И}\,\frac{\vdash \Gamma, A\,[y/x]}{\vdash \Gamma, \text{И}x.A}\,\dagger \qquad \text{Я}\,\frac{\vdash \Gamma, A\,[y/x]}{\vdash \Gamma, \text{Я}x.A}\,\dagger \qquad \text{И-Я}\,\frac{\vdash \Gamma, A\,[y/x]\,, B\,[y/x]}{\vdash \Gamma, \text{И}x.A, \text{Я}x.B}\,\dagger$$

Figure 2: Formulas and sequent rules in PiL.

We then show that the language of π-calculus can be embedded in the language of PiL, and that the linear implication ($\multimap$) captures the reduction semantics. More precisely

$$\begin{aligned} P \to P' \text{ "via Com"} &\Longrightarrow \vdash_{\mathsf{PiL}} [\![P]\!]' \multimap [\![P]\!] \\ P \to P_{\ell_i} \text{ "via the same Choice for each } i\text{"} &\Longrightarrow \vdash_{\mathsf{PiL}} \mathbin{\&}_{i=1}^{n} \left([\![P]\!]_{\ell_i}\right) \multimap [\![P]\!] \end{aligned} \tag{0.3}$$

Moreover, we prove that provability in PiL decides deadlock-freedom. That is, A process P is deadlock-free iff $\vdash_{\mathsf{PiL}} [\![P]\!]$.

The interest in this novel logical framework for the study of safety properties for processes, based on a logic-programming interpretation of proofs (as in [2, 1]) instead of the Curry-Howard interpretation used in session types (see [3, 12]), is that it allows a simpler logical characterization of deadlock-freedom with respect to the one provided by the correspondence "typeable = deadlock-free" in session types which requires either to restrain the language of the process calculus (as in [3, 12, 4]), or to introduce heavy annotations in types (as in [5, 8]).

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Conditionals as quotients in Boolean algebras

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SPEAKER: Francesco Manfucci.

Conditional expressions are central in representing knowledge and reasoning. Conditional reasoning indeed features in a wide range of areas including non-monotonic reasoning, causal inference, learning, and more generally reasoning under uncertainty. A conditional statement is a hypothetical proposition of the form "If [antecedent] is the case, then [consequent] is the case", where the antecedent is assumed to be true. Such a notion can be formalized by expanding the language of classical logic by a binary operator a/b that reads as "a given b". A most well-known approach in this direction is that of Stalnaker [4, 5], further analyzed also by Lewis [2], that in order to axiomatize the operator $/$ ground their investigation on particular Kripke-like structures. In this contribution we base our approach instead in the algebraic framework, in a line of investigation initiated in [1].

The novel approach we propose here is grounded in the algebraic setting of Boolean algebras, where there is a natural way of formalizing conditional statements. Indeed, given a Boolean algebra $\mathbf{B}$ and an element b in B, one can define a new Boolean algebra, say $\mathbf{B}/b$, intuitively obtained by assuming that b is true. More in details, one considers the congruence collapsing b and the truth constant 1, and then $\mathbf{B}/b$ is the corresponding quotient. Then the idea is to define a conditional operator $/$ such that a/b represents the element a as seen in the quotient $\mathbf{B}/b$, mapped back to B. The particular structural properties of Boolean algebras allow us to do so in a natural way.

First, we assume the algebra B to be finite. Then, if $b \neq 0$ the quotient B/b is actually a *retract* of B, which means that if we call π_b the natural epimorphism $\pi_b : B \to B/b$, there is an injective homomorphism $\iota_b : B/b \to B$ such that $\pi_b \circ \iota_b$ is the identity map. The idea is then to consider

$$a/b := \iota_b \circ \pi_b(a). \tag{0.4}$$

We observe that the map ι_b is not uniquely determined, meaning that there can be different injective homomorphisms ι, ι' such that $\pi_b \circ \iota = \pi_b \circ \iota'$ is the identity. Now, in order to be able to define an operator / over the algebra B, one needs to consider all the different quotients, determined by all choices of elements $b \in B$. Then, if $0 \neq b \leq c$, by general algebraic arguments one gets a natural way of looking at nested conditionals; indeed it holds that $(B/c)/\pi_c(b) = B/b$, which means that B/b is a quotient of B/c, and actually also its retract. It is then natural to ask that the choices for ι_b and ι_c be *compatible*, in the sense that there is a way of choosing the embedding $\iota_{\pi_c(b)}$ so that

$$\iota_b = \iota_c \circ \iota_{\pi_c(b)}, \tag{0.5}$$

which yields in particular that $a/b = (a/b)/c$ whenever $b \leq c$.

The case where $b = 0$ needs to be considered separately, since the associated quotient is the trivial algebra that cannot be embedded into B. Since intuitively we are considering the quotients by an element b to mean that "b is true", the *ex falso quodlibet* suggests that we map all elements to 1, i.e:

$$a/0 := 1. \tag{0.6}$$

The idea is then to use Stone duality to translate the above conditions to the dual setting; in other words, we generate the intended models as algebras of sets.

To this end, by the finite version of Stone duality, we now see the algebra B as an algebra of sets, say that $B = (X)$ for a set X. Then the above reasoning translates to the following. Given YX, the natural epimorphism $\pi_Y : (X) \to (Y)$ dualizes to the identity map $_Y : Y \to X$, and the embedding $\iota_Y : (Y) \to (X)$ dualizes to a surjective map $f_Y : X \to Y$, such that $f_Y \circ_Y =_Y$; in other words, we are asking that f_Y restricted to Y is the identity. Moreover, consider YZX. Then the compatibility condition (0.5) becomes on the dual $f_Y = f_Y^Z \circ f_Z$, where f_Y^Z is the dual of the map $\iota_{\pi_Z(Y)}$. The intended models are those that originate by the above postulates; let us be more precise.

Definition 1. Given a set X, we say that a class of surjective functions $F = \{f_Y^Z : Z \to Y : \emptyset \neq YZX\}$ with $f_Y^Z : Z \to Y$ is *compatible with* X if:

1. f_Y^X restricted to Y is the identity on Y;

2. $f_Y^X = f_Y^Z \circ f_Z^X$.

We now define the class of intended models as algebras of sets.

Definition 2. An *intended model* is an algebra with operations $\{\wedge, \vee, \neg, /, 0, 1\}$ that is a Boolean algebra of sets (X) for some set X with / defined as follows from a class of functions F compatible with X:

$$Y/Z := (f_Z^X)^{-1}(Y \cap Z)$$

for any Y, ZX and $Z \neq \emptyset$, and for any YX we set $Y/\emptyset := X$.

We then analyze the algebraic properties of the intended models and the variety they generate, showing also an interesting connection with Stalnaker's approach to conditionals.

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Bi-Intermediate Logics of Co-Trees: Local Finiteness and Decidability

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SPEAKER: Miguel Martins.

Bi-intuitionistic logic bi-IPC is the conservative extension of (propositional) intuitionistic logic IPC obtained by adding a new binary connective $\leftarrow$ to the language, called the *co-implication*, which behaves dually to $\rightarrow$. In this way, bi-IPC reaches a symmetry, which IPC lacks, between the connectives $\wedge, \top, \rightarrow$ and $\vee, \bot, \leftarrow$, respectively. Furthermore, thanks to the co-implication, bi-IPC achieves significantly greater expressivity than IPC. For instance, if the points of a Kripke frame $\mathfrak{M}$ are interpreted as states in time, the language of bi-IPC is expressive enough to talk about the past, something that is not possible in IPC. This feature is captured by the transparent interpretation of co-implication provided by the Kripke semantics of bi-IPC [11], since $\mathfrak{M}, x \models \phi \leftarrow \psi$ iff $\exists y \leq x$ ($\mathfrak{M}, y \models \phi$ and $\mathfrak{M}, y \not\models \psi$).

The greater symmetry of bi-IPC when compared to IPC is reflected in the fact that bi-IPC is algebraized in the sense of [1] by the variety bi-HA of *bi-Heyting algebras* [10], i.e., Heyting algebras whose order duals are also Heyting algebras. As a consequence, the lattice of *bi-intermediate logics* (i.e., consistent axiomatic* extensions of bi-IPC) is dually isomorphic to that of nontrivial varieties of bi-Heyting algebras. The latter, in turn, is not only amenable to the methods of universal algebra, but also from those of duality theory, since the category of bi-Heyting algebras is dually equivalent to that of *bi-Esakia spaces* [5], see also [1].

In [2], we began studying extensions of the *bi-intuitionistic Gödel-Dummett logic* bi-GD := bi-IPC$+(p \rightarrow q) \vee (q \rightarrow p)$, the bi-intermediate logic axiomatized by the *Gödel-Dummett axiom* (also known as the *prelinearity axiom*). Over IPC, this formula axiomatizes the well-known *intuitionistic linear calculus* LC := IPC$+(p \rightarrow q) \vee (q \rightarrow p)$ (see, e.g., [4, 6, 8, 7]). While both logics are Kripke complete with respect to the class of *co-trees* (i.e., posets with a greatest element and whose principal upsets are chains), notably, the properties of these logics diverge significantly. For example, while LC has only countably many extensions, all of which are locally finite, we proved that bi-GD is not locally finite and has continuum many extensions. Moreover, LC is also Kripke complete with respect to the class of chains, whereas we showed that the bi-intermediate logic of chains is a proper extension of bi-GD (namely, the one obtained by adding the *dual Gödel-Dummett axiom* $\neg[(q \leftarrow p) \wedge (p \leftarrow q)]$ to bi-GD). This strongly suggest that the language of bi-IPC is more appropriate to study tree-like structures than that of IPC (since we work with a symmetric language, all of our results can be dualized to the setting of trees in a straightforward manner).

One notable extension of bi-GD is $Log(FC) := \{\varphi \colon \forall n \in \mathbb{Z}^{+} \ (\mathfrak{C}_n \models \varphi)\}$, the *logic of the finite combs* (i.e., finite co-trees whose shape resembles that of a comb, see Figure 1). We showed in [2] that if L is an extension of bi-GD, then L is locally finite iff $L \nsubseteq Log(FC)$. Consequently, $Log(FC)$ is the only pre-locally finite extension of bi-GD (i.e., it is not locally finite, but all of its proper extensions are so). More recently, we found a finite axiomatization for $Log(FC)$, using Jankov and subframe formulas (the theories of these types of formulas for bi-GD were developed in [2, 9]). Since, by definition, this logic has the finite model property, we can conclude that the problem of determining if a recursively axiomatizable extension of bi-GD is locally finite is decidable.

In this talk, we will cover the main steps of our recent proof. Namely, we will provide a characterization of the bi-Esakia duals of the finitely generated subdirectly irreducible algebras which validate bi-GD plus three particular Jankov formulas and one subframe formula. We will then present a combinatorial method we developed which can be used to show that the variety generated by the aforementioned algebras has the finite model property. This allows us to infer that $Log(FC)$ coincides with the extension of bi-GD axiomatized by the above mentioned Jankov and subframe formulas.

*From now on we will use *extension* as a synonym of *axiomatic extension*.

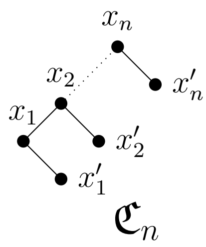

Figure 1: The n-comb $\mathfrak{C}_n$.

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Categorical structures arising from implicative algebras

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SPEAKER: Samuele Maschio.

Implicative algebras have been introduced by Miquel in [3] in order to provide a unifying notion of model, encompassing the most relevant and used ones, such as realizability (both classical and intuitionistic), and forcing. In particular, implicative algebras can be seen both as generalizations of locales and of partial combinatory algebras.

In this talk we will show how various notions can be generalized to implicative algebras, by adopting these perspectives.

By looking at implicative algebras as generalizations of locales, one can express topological concepts in a very wide framework. In this talk we will focus in particular on notions of supercompactness and connectedness.

At the same time, by seeing an implicative algebra as a generalized partial combinatory algebra, one can concentrate on computational concepts and extend them to arbitrary implicative algebras. In particular, we will show that some categorical structures which were introduced in the context of the effective topos [1] and (more in general) of realizability toposes (see e.g. [4]) can be adequately rephrased in the wider context of implicative algebras. In particular, we will abstract the notion of a category of assemblies, partition assemblies, and modest sets to arbitrary implicative algebras, and thoroughly investigate their categorical properties and interrelationships. This work is mainly based on [2].

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O-minimality, domination, and preorders

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SPEAKER: Rosario Mennuni.

I will talk about joint work with P. Andújar Guerrero and P. Eleftheriou on regarding the following question: is it true that, in o-minimal theories, every global invariant type is dominated by the product of finitely many orthogonal 1-types?

I will discuss the background of the problem, its solution in certain special cases, and the connection with recent work of P. Andújar Guerrero, M. Thomas and E. Walsberg on cofinal curves in definable preorders.

Subdirectly irreducible and generic equational states

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SPEAKER: Sebastiano Napolitano.

An Abelian lattice-ordered group (ℓ-group, for short) is an Abelian group G endowed with a lattice order that is translation invariant. An ℓ-group is called unital if it contains an element u, such that for any positive $g \in G$ there exists a natural number n for which the n-fold sum of u exceeds g. A *state* of a unital ℓ-group is a normalized and positive group homomorphism in $\mathbb{R}$. It is well known that states correspond to expected-value operators on bounded real random variables. Unital ℓ-groups are not first-order definable, yet they are categorically equivalent to the equational variety of MV-algebras [1]. Thus, states can be studied in an equational setting by looking at their counterpart in MV-algebras, as first proposed in [9]. However, since states on MV-algebras are defined as particular maps into the real unit interval $[0, 1]$, a completely algebraic characterization was still missing.

Efforts to find an algebraic theory of states continued in [5] (see also [2]). There the authors introduced the notion of *internal state* as an additional unary operation with specific axioms relating it to the other MV-operations. This framework was used to provide an algebraic treatment of the Lebesgue integral. A drawback of this approach is that an internal state can be applied to itself. More recently, a different approach has been proposed. In [6] the authors first extend Mundici's equivalence between unital ℓ-groups and MV-algebras to an equivalence between states between ℓ-groups and states between MV-algebras. Secondly, they introduce the class of *equational states* as a two-sorted variety of algebras. An equational state $(\mathbf{A}_1, \mathbf{A}_2, s)$ is a two-sorted algebra in which each sort $\mathbf{A}_1$ and $\mathbf{A}_2$ is an MV-algebra with customary operations and the *state-operation* s has $\mathbf{A}_1$ as domain and $\mathbf{A}_2$ as codomain. This approach opens the way to studying probabilistic notions with algebraic tools; for instance, [6, Theorem 4.1] gives a characterization of free equational states.

Another reason for considering the class of equational states is that they provide an algebraic semantics to the probabilistic logic FP(Ł, Ł). The system FP(Ł, Ł) is a two-layer logic introduced in [4] to provide a formal framework to deal with the probability of vague events. If a vague event is codified by a formula φ in Łukasiewicz logic, its probability is given by the formula $\Box(\varphi)$, which is a Łukasiewicz atomic formula interpreted as "φ is probable".

An adaptation of the classical Lindenbaum-Tarski construction produces an equational state $\mathcal{ES}_{\mathsf{Var}}$ with the following properties.

Theorem 1 ([7, Theorem 8])**.** *Let* Var *be a (one-sorted) set of propositional variables. For any FP(Ł, Ł) formula* Φ*, the following are equivalent.*

1. Φ *is a theorem.*
2. Φ *is valid in the equational state* $\mathcal{ES}_{\mathsf{Var}}$*.*
3. Φ *is valid in all equational states.*

Corollary 2 ([7, Theorem 15])**.** *The equational state* $\mathcal{ES}_{\mathsf{Var}}$ *is the free equational state generated by* $(\mathsf{Var}, \emptyset)$*.*

We present here a continuation of the algebraic study of equational states started in [8], where it is proven that the lattice of ideals and the lattice of congruences of any equational state are isomorphic (see [8, Corollary 2]). This isomorphism enables us to characterize the subdirectly irreducible equational states as follows.

Theorem 3. *An equational state* $(\mathbf{A}_1, \mathbf{A}_2, s)$ *is subdirectly irreducible if and only if one of the following is true:*

1. $\mathbf{A}_2 = \emptyset$ *and* $\mathbf{A}_1$ *is a subdirectly irreducible MV-algebra.*
2. $\mathbf{A}_2$ *is a subdirectly irreducible MV-algebra, and the state-operation is faithful, i.e.* $s(x) = 0$ *implies* $x = 0$*.*

Combining the characterization of subdirectly irreducible equational states with some ideas of [3] we prove that two notable classes generate the variety of equational classes.

Theorem 4. *The following classes generate the variety of equational states:*

1. *The class of all equational states of the type* $([0,1]^W, [0,1])$*, with* W *an arbitrary set.*
2. *The class of finite equational states, i.e. equational states whose universe is finite in each sort.*

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The Galvin-Prikry Theorem in the Weihrauch lattice

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SPEAKER: Gian Marco Osso.

I will address the classification of different fragments of the Galvin-Prikry theorem, an infinite dimensional generalization of Ramsey's theorem, in terms of their uniform computational content (Weihrauch degree). This work can be seen as a continuation of [1], which focused on the Weihrauch classification of functions related to the Nash-Williams theorem, i.e., the restriction of the Galvin-Prikry theorem to open sets. We have shown that most of the functions related to the Galvin-Prikry theorem for Borel sets of rank n are strictly between the $(n+1)$-th and n-th iterate of the hyperjump operator HJ, which corresponds to the system Π^1_1-CA_0 in the Weihrauch lattice. To establish this classification we obtain the following computability theoretic result (along the lines of [2] and [3]): a Turing jump ideal containing homogeneous sets for all $\Delta^0_{n+1}(X)$ sets must also contain $\mathsf{HJ}^n(X)$. Similar results also hold for Borel sets of transfinite rank. This is joint work with Alberto Marcone.

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Kanger-Wang-type Sequent Calculi with Equality

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SPEAKER: Franco Parlamento.

On the ground of the general result in [5], we develop a proof theoretic analysis of several extensions of the Dragalin's $\mathbf{G3}[\mathbf{mic}]$ sequent calculi ([1]) obtained by adding rules for equality related to those introduced by H.Wang ([6]) and S.Kanger ([3]). In the classical case we apply our results to the semantic tableau method for first order logic with equality. In particular we establish that, for languages without function symbols, in Fitting's alternate semantic tableau method ([2]), strictness (which does not allow to retain the formulae that are modified except for γ-formulae of course) can be imposed together with the orientation of the replacement of equals provided the latter is allowed on all atomic formulae and their negations. Furthermore we prove that the result holds also for languages with function symbols provided strictness is not imposed on equalities, leaving it open whether or not strictness can be imposed on equalities as well. Finally we discuss to what extent the strengthened form of the nonlengthening property of Orevkov known to hold for the sequent calculi with the structural rules ([4]) applies also to the present context.

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Geometry of interaction and non-determinism

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SPEAKER: Marco Pedicini.

Lamping's optimal algorithm is a graph-theoretic technique for the normalization of λ-terms whereby a redex is never duplicated, so as to keep as low as possible the number of beta-reduction steps. Although optimality for standard, deterministic λ-calculus has been well understood through a large body of research, the situation is quite different for the notion of optimality of non-deterministic λ-calculus in which the non duplication constraint becomes much more intricate and subtle.

We investigate an open and challenging conceptual issue in the territory of non-deterministic λ-calculus, namely the complete description of an optimal reduction strategy for it. The adequacy of such description also raises some prospect of a sound interpretation of quantitative execution in probabilistic programming languages along the lines of the parallel model of execution. Optimal reduction for standard λ-calculus was defined by Lévy in 1978 and implemented by Lamping at the end of eighties [Lév78], [Lam89]. It turned to be connected to the Geometry of Interaction introduced by J.-Y. Girard to afford a semantic picture of linear logic and to model the dynamics of cut-elimination algorithm via a mathematical, syntax-free characterization of it [Gir89], [DR93]. We design an optimal reduction algorithm for non-deterministic λ-calculus and we establish its soundness in the geometry of interaction setting. Moreover, we combine optimal reduction for non deterministic λ-calculus with parallel execution as considered in [PQ07] and in [LPP19].

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Constructive Stone representations for separated swap and Boolean algebras

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SPEAKER: Iosif Petrakis.

The theory of swap algebras and swap rings is a generalisation of the theory of Boolean algebras and Boolean rings that originates from Bishop-style constructive mathematics (BISH) (see [1, 4]) and has non-trivial applications to classical mathematics (CLASS) (see [3, 5]). Swap algebras and swap rings are introduced in [3] as abstract versions of the class of complemented subsets and partial Boolean-valued functions, respectively. Complemented subsets i.e., pairs of subsets which are disjoint in a strong sense, were introduced by Bishop in [1] as an important tool in his constructive reconstruction of measure and integration theory. As the powerset $\mathcal{P}(X)$ of a set X is classically bijective to the total Boolean-valued functions on X, the complemented power set $\mathcal{P}^{)(}(X)$ of X is constructively bijective to the partial Boolean-valued functions on X (see [6]). A Boolean algebra and a Boolean ring is a special case of a swap algebra and a swap ring, respectively, and in [3] it is shown that the duality between swap algebras of type* (II) and swap rings generalises the duality between Boolean algebras and Boolean rings.

Here we extend our results in [3] presenting a constructive Stone representation theorem for separated swap algebras of type (II) that has as a special case a constructive Stone representation theorem for separated Boolean algebras.

There are two problems in the constructivisation of Stone representation theorem for Boolean algebras. The first problem is the use of "points". For example, if one wants to define pointed Boolean-valued homomorphisms on $\mathcal{P}(X)$, then one needs to use the principle of the excluded middle (PEM); if $x_0 \in X$, then $\widehat{x_0} \colon \mathcal{P}(X) \to \mathbf{2}$, where $\mathbf{2} := \{0, 1\}$, is defined with PEM by the rule

$$\widehat{x_0}(A) := \begin{cases} 1, & x_0 \in A \\ 0, & x_0 \notin A. \end{cases}$$

It is with these Boolean-valued homomorphisms that one shows that $\mathcal{P}(X)$ is separated. We call a Boolean algebra *separated*, if for every $a \neq 0$ there is a Boolean-valued homomorphism h_a on A with $h_a(a) = 1$. If $A \neq \emptyset$, then classically there is a point $x_0 \in A$, hence $\widehat{x_0}(A) = 1$. If $\boldsymbol{A} := (A^1, A^0)$ is a complemented subset of set X with an extensional inequality, where A^1, A^0 are disjoint in a strong sense (see [3] for all details), then one can define the above pointed Boolean-valued homomorphism, not on the whole complemented powerset, but on its subset of all complemented subsets $\boldsymbol{A}$ for which their domain $A^1 \cup A^0$ contains x_0, using the following rule and avoiding PEM

$$\widehat{x_0}(\boldsymbol{A}) := \begin{cases} 1, & x_0 \in A^1 \\ 0, & x_0 \in A^0. \end{cases}$$

With these partial, pointed Boolean-valued homomorphisms we show constructively that $\mathcal{P}^{)(}(X)$ is a separated swap algebra of type (II), a notion that generalises the notion of a separated Boolean algebra and involves the set $\widehat{A}$ of partial, Boolean-valued swap-homomoprhisms of a swap algebra A of type (II). The following representation theorem is shown constructively.

Theorem 1 (Stone representation theorem for separated swap algebras of type (II))**.** *If A is a separated swap algebra of type* (II)*, then the assignment routine* $\texttt{Stone} \colon A \rightsquigarrow \mathcal{P}^{)(}(\widehat{A})$*, where* $a \mapsto \texttt{Stone}(a)$*, with*

$$\texttt{Stone}(a) := \big(\texttt{Stone}^1(a), \texttt{Stone}^0(a)\big),$$

$$\texttt{Stone}^1(a) := \{f \in \widehat{A} \mid a \in \mathrm{dom}(f) \wedge f(a) =_{\mathbf{2}} 1\},$$

and

$$\texttt{Stone}^0(a) := \{g \in \widehat{A} \mid a \in \mathrm{dom}(g) \wedge g(a) =_{\mathbf{2}} 0\},$$

is a total swap-embedding of A into the swap algebra $\mathcal{P}^{)(}(\widehat{A})$ of type (II).

The second problem in the constructivisation of the Stone representation theorem is that one needs Zorn's lemma to show that every Boolean algebra is separated (see [2],

*There are two types of swap algebras, as there are two algebras of complemented subsets i.e., two ways to define their join and meet (see [6, 3]).

p. 77). What we show here is that we do not need to employ some maximal object, only to change the equality of A. In topology a similar attitude is followed in the theory of the ring of continuous functions $C(X)$ of a topological space X (see [7]), where from the point of view of $C(X)$ it suffices to restrict to completely regular topological spaces. Similalrly, from the point of view of the theory of $\widehat{A}$, it suffices to restrict to separated swap algebras. We denote by $\widehat{(A, B)}$ the set of partial swap-homomorphisms between swap algebras A and B.

Theorem 2 (Stone-Čech theorem for swap algebras of type (II))**.** *If $(A, =_A, \neq_A)$ is a swap algebra of type* (II)*, there is a separated swap algebra of type* (II) $(\sigma A, =_{\sigma A}, \neq_{\sigma A})$ *and a total swap-homomorphism $\sigma_A \colon A \to \sigma A$, such that for every $f \in \widehat{A}$ there is a unique $\sigma f \in \widehat{\sigma A}$, such that the following left triangle commutes i.e.,* $\mathrm{dom}(\sigma f) := \{\sigma_A(a) \mid a \in \mathrm{dom}(f)\}$ *and* $\sigma f(\sigma_A(a)) =_{\mathbf{2}} f(a)$*, for every* $a \in \mathrm{dom}(f)$.

$$\begin{array}{ccc} A & \xrightarrow{\sigma_A} & \sigma A \\ & \widehat{A} \ni f \searrow & \downarrow \sigma f \in \widehat{\sigma A} \\ & & \mathbf{2} \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\sigma_A} & \sigma A \\ & \widehat{(A, B)} \ni h \searrow & \downarrow \sigma h \in \widehat{(\sigma A, B)} \\ & & B \end{array}$$

Also, for every $h \in \widehat{\sigma A}$ there is a unique $h' \in \widehat{A}$, such that $\sigma h' =_{\widehat{\sigma A}} h$, and hence $\widehat{A} =_{\mathbb{V}_0} \widehat{\sigma A}$ i.e., the two sets are bijective. If $(B, =_B, \neq_B)$ is a separated swap algebra of type (II)*, and $h \colon A \rightharpoonup B$ is a partial swap-homomorphism, there is a unique partial swap-homomoprhism $\sigma h \colon \sigma A \rightharpoonup B$, such that the above right triangle commutes i.e.,* $\mathrm{dom}(\sigma h) := \{\sigma_A(a) \mid a \in \mathrm{dom}(h)\}$ *and* $\sigma h(\sigma_A(a)) =_{\mathbf{2}} h(a)$*, for every* $a \in \mathrm{dom}(h)$*. Also, for every $h \in \widehat{(\sigma A, B)}$ there is a unique $h' \in \widehat{(A, B)}$, such that $\sigma h' =_{\widehat{(\sigma A, B)}} h$, and hence $\widehat{(A, B)} =_{\mathbb{V}_0} \widehat{(\sigma A, B)}$.*

We also present the corresponding Stone representation theorem for separated Boolean algebras and the Stone-Čech theorem for Boolean algebras.

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Some results in non-monotonic proof-theoretic semantics

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SPEAKER: Antonio Piccolomini d'Aragona.

Proof-theoretic semantics (henceforth, PTS) is a constructive, proof-based semantics stemming from Prawitz's normalisation results for natural deduction. In the original formulation of PTS, due to Prawitz's himself [4], proofs are understood as *valid arguments*. An *argument* is a pair $\langle \mathscr{D}, \mathfrak{J} \rangle$ such that

- $\mathscr{D}$ is a formula-tree of arbitrary inferences (that may bind top-nodes). Unbound top-nodes Γ are the *open assumptions* of $\mathscr{D}$, while the root A is the *conclusion* of $\mathscr{D}$. $\mathscr{D}$ is also called an *argument structure* from Γ for A. $\mathscr{D}$ is *open* when $\Gamma \neq \emptyset$, while it is *closed* otherwise. $\mathscr{D}$ is *canonical* when it ends by introduction, while it is *non-canonical* otherwise. A (*closed*) *instance* of $\mathscr{D}$ is a (closed) $\mathscr{D}^\sigma$ resulting from $\mathscr{D}$ through a substitution σ which replaces the open assumptions of $\mathscr{D}$ with (closed) argument structures for those assumptions;

- $\mathfrak{J}$ is a set of functions ϕ from and to argument structures such that, if $\mathscr{D}$ is from Γ for A, $\phi(\mathscr{D})$ is from $\Gamma^* \subseteq \Gamma$ for A and, for every σ, $\phi(\mathscr{D}^\sigma) = \phi(\mathscr{D})^\sigma$.

Argumental validity is relativised to *atomic bases* $\mathfrak{B}$, i.e., sets of atomic rules of level k of the form $([\Gamma_1]/A_1, ..., [\Gamma_n]/A_n)/B$, where A_i, B are atomic and Γ_i/A_i is a rule of level $h < k$—square brackets indicate binding. In the non-monotonic approach, the PTS-*validity* of $\langle \mathscr{D}, \mathfrak{J} \rangle$ *on* $\mathfrak{B}$ is defined as follows:

1) if $\mathscr{D}$ is closed $\Longrightarrow$ by suitably applying the functions in $\mathfrak{J}$, $\mathscr{D}$ yields a canonical argument structure whose immediate sub-structures are valid on $\mathfrak{B}$ when paired with $\mathfrak{J}$;

2) if $\mathscr{D}$ is open from assumptions $A_1, ..., A_n \Longrightarrow$ for every closed $\mathscr{D}_i^\sigma$ for A_i which is valid on $\mathfrak{B}$ when paired with an extension $\mathfrak{J}^+$ of $\mathfrak{J}$, the corresponding closed instance of $\mathscr{D}$ is valid on $\mathfrak{B}$ when paired with $\mathfrak{J}^+$.

This is non-monotonic as there are $\langle \mathscr{D}, \mathfrak{J} \rangle$ and $\mathfrak{B}$ with $\langle \mathscr{D}, \mathfrak{J} \rangle$ valid on $\mathfrak{B}$ and, for some extension $\mathfrak{B}^+$ of $\mathfrak{B}$, $\langle \mathscr{D}, \mathfrak{J} \rangle$ not valid on $\mathfrak{B}^+$. The monotonic definition obtains through a change in clause 2), by requiring the $\mathscr{D}_i^\sigma$-s to be valid on arbitrary extensions of $\mathfrak{B}$. $\langle \mathscr{D}, \mathfrak{J} \rangle$ is *valid* iff it is valid on all $\mathfrak{B}$-s. We say that $\Gamma \models_{\mathfrak{B}}^{\alpha} A$ iff there is a $\mathfrak{B}$-valid $\langle \mathscr{D}, \mathfrak{J} \rangle$ where $\mathscr{D}$ is from Γ for A, and that $\Gamma \models^{\alpha} A$ iff there is a valid $\langle \mathscr{D}, \mathfrak{J} \rangle$ with $\mathscr{D}$ as above.

In the current mainstream approach to PTS, argument structures and reductions are left out. This yields a sentential semantics, due to de Campos Sanz, Piecha and Schroeder-Heister [2, 3], where $\Gamma \models_{\mathfrak{B}} A$ holds iff:

a) $\Gamma = \emptyset \Longrightarrow$

i) A atomic $\Longrightarrow A$ theorem in $\mathfrak{B}$;

ii) $A = B \wedge C \Longrightarrow \models_{\mathfrak{B}} B$ and $\models_{\mathfrak{B}} C$;

iii) $A = B \vee C \Longrightarrow \models_{\mathfrak{B}} B$ or $\models_{\mathfrak{B}} C$;

iv) $A = B \to C \Longrightarrow B \models_{\mathfrak{B}} C$;

b) $\Gamma \neq \emptyset \Longrightarrow (\models_{\mathfrak{B}} \Gamma \Longrightarrow \models_{\mathfrak{B}} A)$.

This is again non-monotonic. The corresponding monotonic version obtains by modifying clause b) so as to bring expansions of $\mathfrak{B}$ in. A variant due to Sandqvist [5], which we may indicate by $\Gamma \models^S_{\mathfrak{B}} A$, defines the disjunction case in an elimination-like way, i.e., it replaces clause iii) by

iii)* $A = B \vee C \Longrightarrow$ for every atomic D, $(B \models^S_{\mathfrak{B}} D$ and $C \models^S_{\mathfrak{B}} D \Longrightarrow \models^S_{\mathfrak{B}} D)$.

Finally we say that $\Gamma \models A$ [resp. $\Gamma \models^S A$] iff, for every $\mathfrak{B}$, $\Gamma \models_{\mathfrak{B}} A$ [resp. $\Gamma \models^S_{\mathfrak{B}} A$]. Here, the interest of Sandqvist's variant lies in the fact that intuitionistic logic is complete relative to the monotonic version of it when atomic bases have level ≥ 2 [5]—while it is incomplete relative to monotonic $\models$, and to non-monotonic $\models$ if atomic bases have unlimited level [2, 3].

The relations among the three consequence-notions $\models^\alpha, \models$ and $\models^S$ have been little explored so far. This seems to be an interesting topic, though. $\models^\alpha$ may be seen as a kind of "witnessing", via suitable argument structures, of the holding of $\models$ and $\models^S$. In [1], I investigated this relative to monotonic PTS, and proved some inversion results which have relevant consequences relative to the completeness and incompleteness results for $\models$ or $\models^S$ mentioned above. Now I aim instead to discuss how those results extend to the non-monotonic framework.

In particular I show that, if one has a sufficiently "loose" notion of function for manipulating argument structures, $\models$ and $\models^\alpha$ are equivalent, both locally—i.e., $\Gamma \models_{\mathfrak{B}} A \Longleftrightarrow \Gamma \models^\alpha_{\mathfrak{B}} A$—and globally—i.e., $\Gamma \models A \Longleftrightarrow \Gamma \models^\alpha A$. In the case of Sandqvist's variant, one can instead prove an inversion result such that, if $\models^S_{\mathfrak{B}}$ is always witnessed by $\models^\alpha_{\mathfrak{B}}$, one can always extract a Sandqvist consequence from a given witnessing—namely if, for all Γ and A, $\Gamma \models^S_{\mathfrak{B}} A \Longrightarrow \Gamma \models^\alpha_{\mathfrak{B}} A$, then for all Γ and A, $\Gamma \models^\alpha_{\mathfrak{B}} A \Longrightarrow \Gamma \models^S_{\mathfrak{B}} A$. At the global level, we can give a sufficient condition, through notions of base-soundness and base-completeness: a logic Σ is *base-complete* [resp. *base-sound*] on a proof-theoretic consequence relation $\Vdash$ iff, for all Γ, A and $\mathfrak{B}$, $\Gamma \Vdash_{\mathfrak{B}} A \Longrightarrow \Gamma \vdash_{\Sigma \cup \mathfrak{B}}$ [resp. $\Gamma \vdash_{\Sigma \cup \mathfrak{B}} A \Longrightarrow \Gamma \Vdash_{\mathfrak{B}} A$] [1]. Thus we obtain that $\Gamma \models^S A \Longleftrightarrow \Gamma \models^\alpha A$ is implied by the existence of a Σ which be base-complete on $\models^S$ and base-sound on $\models^\alpha$. While the latter is also a sufficient condition for the simple completeness of Σ on $\models^\alpha$, it can be proved that Σ cannot be base-complete on any $\Vdash$ if it enjoys the Export Principle—see [3] for the definition of the Export Principle.

In a stricter reading of functions for operating on argument structures, the results we obtain come to depend on the meta-logic we are using. With intuitionistic logic, $\models$ and $\models^\alpha$ may not be equivalent, but we have inversion results similar to those for $\models^S$ and $\models^\alpha$. With classical logic, $\models^\alpha$ and $\models$ are still locally equivalent. However, if we allow loose functions again, classical logic can be proved to be sound and complete with respect to both $\models$ and $\models^{\alpha}$—although under a stricter reading only completeness under substitutions will hold. Interestingly, this stronger outcome may fail when functions are understood in a suitably strict way (despite the meta-logic being classical).

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Generalized Baire Class Functions

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SPEAKER: Beatrice Pitton.

Generalized descriptive set theory (GDST) aims at developing a higher analogue of classical descriptive set theory in which ω is replaced with an uncountable cardinal κ in all definitions and relevant notions. In this talk, we work with functions $f\colon X \to Y$ between κ-metrizable spaces X and Y of weight at most κ, where κ satisfies $\kappa^{<\kappa} = \kappa > \omega$. In this context, we show that the κ^+-Borel functions are the smallest collection containing the continuous functions and closed under $\leq \kappa$-pointwise limits. We introduce the definition of κ-Baire class ξ functions for $1 \leq \xi < \kappa^+$ and, mirroring the well-known characterization in classical descriptive set theory, we highlight the link between κ-Baire class ξ functions and $\mathbf{\Sigma}^0_{\xi+1}(\kappa^+)$-measurable functions. This result follows from a stronger theorem that characterizes Baire class 1 functions as $\leq \kappa$-pointwise limits of full functions, which are exceptionally simple Lipschitz functions. Finally, as in classical descriptive set theory, we characterize continuous functions and Baire class 1 functions through games.

A Stone duality for the class of compact Hausdorff spaces

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SPEAKER: Elena Pozzan.

We present an algebraic characterization of T_0-topological spaces in terms of preorders describing a base for the space. In particular, we show that any T_0-topological space can be represented as the space whose points are the neighborhood filters of one of its basis for the open sets. Conversely, we show that any dense family of filters on a preorder defines a topological space whose characteristics are strictly connected to the ones of the preorder. Therefore, we show how the separation properties of the topological space can be described in terms of the algebraic properties of the corresponding preorder and family of filters.
Furthermore, drawing on Orrin Frink's article [1], we outline the algebraic conditions on a selected base of the topological space ensuring that the space is compact and Hausdorff. In his article, Frink proved that a space is Tychonoff if and only if it admits a *normal* base for the closed sets of a space. If Z is a normal base for X, then the space of Z*-ultrafilters* forms a Hausdorff compactification for X.
Following this approach, we show that every Tychonoff space can be described as the space whose points are *some* minimal prime filters of a particular type of distributive lattices. Furthermore, we show that the space obtained considering *all* the prime minimal filters of it forms a Hausdorff compactification of the original Tychonoff space.
These results allows us to define a duality between the category of compact Hausdorff spaces with continuous maps and a suitable category of lattices.
This is joint work with Matteo Viale.

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Central sets and infinite monochromatic exponential patterns

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SPEAKER: Mariaclara Ragosta.

Schur's Theorem (1916) states that for every finite coloring of $\mathbb{N}$ there exists a monochromatic triple $a, b, a+b$. Several decades later, Folkman extended this statement by including in a same color arbitrarily long sequences and all finite sums from them. A breakthrough was made in 1974 by Hindman, who showed, in the same setting, the existence of a unique infinite sequence such that all finite sums are monochromatic, and one year later the theorem was extended to all associative operations.

In this talk we explore the case of exponentiation, first investigated by Sisto (2011) and recently by Sahasrabudhe (2018). The latter proved a version of Folkman's Theorem for product and exponentiation at the same time. In our main theorem we realise for the exponentiation the passage from finite to infinite made by Hindman for the sum, by showing that for every finite coloring of $\mathbb{N}$ there exists an infinite sequence such that all finite exponentiations are monochromatic.

We finally extend the theorem to a larger class of binary non-associative operations which somehow behave in the same manner as exponentiation.

Our main tool is a very special type of ultrafilter, idempotent and minimal at the same time. Elements of these ultrafilters, the so-called central sets, contain plenty of combinatorial structure.

This is joint work with Mauro Di Nasso.

Correctness Criterion for Second Order Multiplicative Linear Logic

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SPEAKER: Adrien Ragot.

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Correctness of Proof structures. Linear Logic was introduced by J.Y. Girard in 1987 together with the *proof–structures* – that we call nets and formalize as hypergraphs – graph like objects which may or may not represent a proof. Formally, one can map proof trees from the sequent calculus to proof–structures: this map is called the *desequentialization*, proof structures which are the result of desequentialization are called *proof nets* or are said to be *correct*. From a computational point of view the correctness of a net corresponds to verifying that it is correctly typed: a correct net which is typeable A is a net which behaves as described by the formula A – in the sense of Realisability for Linear Logic see [8] – for the multiplicative fragment one can look at [7]. Providing a procedure which identifies if a net is correct is therefore a form of *verification* ensuring that the net behaves as described by the (typing) formula under consideration. Such procedures are *correctness criteria*.

Correctness for MLL_2. Many criteria exist for distinct fragments of linear logic such as the long–trip criterion [5], Danos–Regnier (DR) criterion [4], contractibility [3], parsing [1], or the counter proof criterion [2]. All these criteria follow a similar pattern: they can be expressed for the fragment MLL and be extended to (some, if not all) other fragments of LL (with more or less difficulty). In particular, we will provide correctness criteria for the fragment $\mathsf{MLL}_2{}^\star$, this fragment is less fortunate than others: currently, the only existing criterion is that of J.Y. Girard given in [6], an adaptation of the Danos–Regnier criterion to MLL_2. We observe several downfalls of the criterion of J.Y. Girard and propose solutions by providing two novel criterions – a simpler adaptation of the Danos–Regnier criterion and a parsing criterion. The limitations of the currently existing criterion from [6] are the following:

1. The nets are supposed typed – we will provide a criterion for untyped nets.
2. The conclusion of the nets must be closed formulas, it will turn out that this restriction is not necessary in order to test correctness.
3. The desequentialization is difficult, without surprise the Danos–Regnier is not the right tool to extract a proof from a net – to that end we will provide a parsing criterion which can easily extract a proof from a parsing sequence,
4. The complexity is exponential (as the Danos Regnier criterion for MLL), and to make it worse: the pointers of a $\forall$–link quantifying a propositional variable X point at *all* the vertex where X occurs and so the number of switchings explodes – we will identify a minimal set of pointers which is enough to ensure correctness, furthermore the parsing criterion will enjoy a much better complexity (quadratic–time).

Parsing criterion for MLL_2. Technically, the first criterion we provide is the parsing criterion from which we will deduce the DR–style criterion. We follow the spirit of the solution proposed by Banach in [1] in particular we use generalised axioms (that we write ✠), and the parsing rules corresponds to rewriting that take place at the top of net rewriting one or several generalised axioms in a single one. The criterion is following: a net can be rewritten into a single generalised axiom if and only if it is correct. To reconstruct a proof we annotate the generalised axioms with proofs from the sequent calculus then the proof consists in: (1) Showing that for each parsing reduction the new daimon is still annotated by a proof. (2) Show that the normal forms for anti–parsing of a single generalised axiom annotated by a proof π is $[\![\pi]\!]$ which the desequentialization of π. (3) Noting that each parsing redex is the desequentialization of a proof hence establishing that each correct net can be contracted.

*that is Multiplicative Linear Logic with second order quantifiers.

DR–style criterion. From the parsing criterion one can easily derive a DR–style criterion. First we need to define the switchings: the $\forall$ links point on the output of generalised axioms which contains the variable X, we make $\forall$ links switch between one of their pointers or their main input. However, the pointers must be *rerouted* to test correctness: a $\forall$ pointer go down the net until it encounters an $\exists$ pointer p then the $\forall$ pointer is rerouted so that it now targets the source of p. Rerouting all the $\forall$–pointers and then performing the switching of $\forall$ links and $\parr$ links yields a switching of the net. The theorem is then the following: a net is correct if and only if all its switchings are acyclic and connected (ACC). To prove this result we follow the idea proposed by Curien in [2]: (1) We show that each parsing rule and anti parsing rule preserves the ACC property of switchings. (2) We show that the only correct nets which are normal for the parsing rewriting are nets made of only one generalised axiom.

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Valued difference fields beyond surjectivity

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SPEAKER: Simone Ramello.

Understanding valued fields has historically been a game of finding the correct invariants for their theories, reducing the possibly complicated valuational structure to something easier to understand and classify. Given a valued field (K, v), one can naturally look at its value group Γ_K and residue field k as possible (model-theoretic) invariants, in the sense that one could hope to determine the theory of (K, v) using the theories of Γ_K and k. This concretized in the celebrated Ax-Kochen/Ershov results. A great industry has since then blossomed, which seeks to generalize similar results in various directions. We tackle one such possible way of making the question harder, namely we enrich the structure of (K, v) with a valued field endomorphism σ. Under the assumption that σ is surjective and that k has characteristic zero, Durhan and Onay ([3]) prove that quantifiers can be reduced to the leading term structure RV_K of the valued field (K, v), when endowed with the induced automorphism σ_{RV}.

At the same time, a considerable bulk of work has been produced by Dor and Hrushovski ([1]), and later Dor and Halevi ([2]), to tackle the *absolute* version of these results, namely identifying the model companion for the so-called ω-increasing case. Here, the assumption of residue characteristic zero is dropped, leading to the richer structure (and hence, harder problems) of twists of the automorphism produced by composing with Frobenius.

Our work uses techniques and insights from the ω-increasing world to remove the assumption that σ is surjective from the work of Durhan and Onay (while we maintain the assumption that σ_{RV} is surjective).

Theorem 1. *Suppose (K, v, σ) is a weakly transformally henselian valued difference field in equicharacteristic zero, such that $\sigma(K)$ is relatively algebraically closed in K. Then, in the natural two-sorted language enriched with λ-functions for K as a $\sigma(K)$-vector space, (K, v, σ) eliminates quantifiers down to $(\mathrm{RV}_K, \sigma_{\mathrm{RV}})$.*

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Ultracompletions

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SPEAKER: Giuseppe Rosolini.

The notion of ultracategory was introduced by Michael Makkai in [8] for the characterisation of categories of models of pretoposes, an ample extension to (intuitionistic) first order theories of Stone duality for Boolean algebras, providing a kind of Stone duality for first order theories—aka conceptual completeness. Recently, Jacob Lurie refined that notion in [7] producing another approach to the duality for pretoposes—the two notions of ultracategory appear to be different, though no separating example has been produced yet. All this suggests that there are already two forms of duality for first order theories, in line with Esakia duality [4] as well as others, see [1, 2].

A excellent, radically new, approach to ultrafilters, ultraproducts, ultracategories, and pretoposes can be found in [5] where the author also foresees a possible comparison of the two original notions of ultracategories in future work.

In this work, we produce an algebraic notion of structured category which subsumes the two kinds of ultracategories mentioned above. Technically, we introduce a colax idempotent pseudomonad on an ***ultracompletion*** 2-functor on the 2-category $\mathbf{Cat}$ of small categories. Given a (small) category C, write $\mathfrak{U}(C)$ for the category which consists of following data:

Objects are triples $(I, \mathcal{U}, (c_i)_{i\in I})$ where $\mathcal{U}$ is an ultrafilter on the set I, and $(c_i)_{i\in I}$ is an I-indexed family of objects in C.

An arrow $[V, f, (g_v)_{v\in V}] : (I, \mathcal{U}, (c_i)_{i\in I}) \to (J, \mathcal{V}, (d_j)_{j\in J})$ is represented by a triple of a set $V \in \mathcal{V}$, a function $f : V \to I$ such that the inverse image of a set in $\mathcal{U}$ is a set in $\mathcal{V}^\star$, and a family $(g_v : c_{f(v)} \to d_v)_{v\in V}$ of arrows in C. Two representatives $(U, f, (g_v)_{v\in V})$ and $(U', f', (g'_v)_{v\in V'})$ are equivalent if $g_v = g'_v$ for all $v \in V \cap V'$.

Composition of arrows is given componentwise.

Remark. Let T denote a terminal category. The ultracompletion $\mathfrak{U}(T)$ is (equivalent to) the opposite of the category UF of ultrafilters of [5]. More generally, $\mathfrak{U}(C)$ is equivalent to $\left(UF_{\mathrm{Fam}(C^{\mathrm{op}})}\right)^{\mathrm{op}}$, where Fam is the usual coproduct completion of a category.

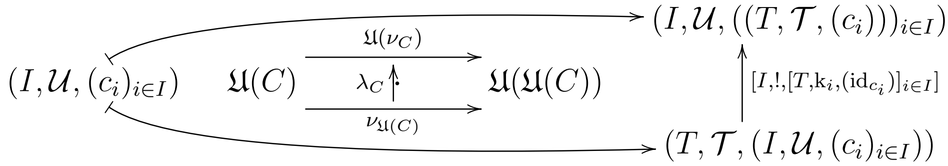
The assignment $C \longmapsto \mathfrak{U}(C)$ extends to a 2-functor $\mathfrak{U} : \mathbf{Cat} \longrightarrow \mathbf{Cat}$, which we call ***ultracompletion***.

In the talk, we shall amply justify the need for this kind of technical details. Here, for sake of completeness, we only introduce the rest of the structure on the ultracompletion functor (write T for a fixed one-element set): for a fixed category C, the unit functor $\nu_C : C \longrightarrow \mathfrak{U}(C)$ takes an object c to the triple $(T, \{T\}, (c))$ consisting of a one-object family. The multiplication functor

$$\begin{array}{ccc} \mathfrak{U}(\mathfrak{U}(C)) & \xrightarrow{\qquad \mu_C \qquad} & \mathfrak{U}(C) \\ (I, \mathscr{U}, (J_i, \mathscr{V}_i, (c_j)_{j\in J_i})_{i\in I}) & \longmapsto & (\sum_{i\in I} J_i, \sum_{\mathscr{U}} \mathscr{V}_i, (c_{(i,j)})_{(i,j)\in\sum_{i\in I} J_i}) \end{array}$$

*In other words, $f^{-1} : \wp(I) \to \wp(J)$ maps $\mathcal{U} \subseteq \wp(I)$ into $\mathcal{V} \subseteq \wp(J)$, see [5].

which employs the indexed sum of ultrafilters, see [5]. It is easy to see that they provide the data for a pseudomonad U on Cat. Finally we introduce a natural family of natural transformations



where $\mathrm{k}_i : T \to I$ is the constant function with value i.

Theorem. *The quadruple* $\mathrm{U} := (\mathfrak{U}, \mu, \nu, \lambda)$ *is a colax idempotent pseudomonad on* $\mathbf{Cat}$.

The ultracompletion functor can be connected with both notions of ultracategories. For the sake of clarity, we shall denote by M-$\mathbf{Ultcat}$, the 2-category of ultracategories, ultrafunctors, and natural ultra-transformations in the sense of Makkai's [8], and by L-$\mathbf{Ultcat}$, the 2-category of ultracategories, ultrafunctors, and natural ultra-transformations in the sense of Lurie's [7].

Proposition. *Let C be a category.*
(i) The category $\mathfrak{U}(C)$ is an ultracategory in the sense of Makkai, and the 2-functor $\mathfrak{U}$: $\mathbf{Cat} \longrightarrow \mathbf{Cat}$ factors through the forgetful 2-functor M-$\mathbf{Ultcat} \longrightarrow \mathbf{Cat}$.
(ii) The category $\mathfrak{U}(C)$ is an ultracategory in the sense of Lurie, and the 2-functor $\mathfrak{U}$: $\mathbf{Cat} \longrightarrow \mathbf{Cat}$ factors through the forgetful 2-functor L-$\mathbf{Ultcat} \longrightarrow \mathbf{Cat}$.

Corollary. *Each* U*-pseudoalgebra $\mathfrak{U}(C) \xrightarrow{\alpha} C$ bears a structure of ultracategory in the sense of Makkai, and a structure of ultracategory in the sense of Lurie, in such ways that each assignment extends to a faithful 2-functor from* U-$\mathbf{PsAlg}$ *into* M-$\mathbf{Ultcat}$ *and into* L-$\mathbf{Ultcat}$*, respectively.*

Finally, we have a result along the lines of Theorem 4.1 of [8].

Theorem. *Let P be a pretopos. The evaluation functor* $\mathrm{ev} : P \longrightarrow \mathrm{U}(\mathrm{PreTop}(P, \mathbf{Set}), \mathbf{Set})$ *is an equivalence of categories.*

The next steps will consider more closely the relationship between U-pseudoalgebras and ultracategories in the sense of Makkai, the connections with the work of Garner's in [5], and the abstract part of duality in line with previous work as in [3, 6, 9].

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A topological reading of (co)inductive definitions

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SPEAKER: Pietro Sabelli.

In predicative and constructive mathematics, new objects must often be constructed via inductive or coinductive definitions to avoid circular definitions. In particular, this applies in the predicative and constructive development of topology over a type-theoretic foundation known as Formal Topology, ideated by P. Martin-Löf and G. Sambin [11, 12].

In Formal Topology, the main object of study is a point-free notion of topological space in which two relations, called *basic covers* and *positivity relations*, primitively represent its open and closed subsets, respectively. To represent many natural topologies such as Cantor topology, Baire topology, or the real numbers line, powerful techniques for inductively generating basic covers and coinductively generating positivity relations have been developed in [3, 13] and have since been a cornerstone of the field.

The (co)inductive methods of Formal Topology have been studied in [6] in the framework of the Minimalist Foundation, a foundational theory for predicative and constructive mathematics conceived in [8] and fully formalised in [5], which is compatible with all the most relevant foundations for mathematics. In particular, it has been shown that the Minimalist Foundation extended with such methods can be interpreted using a realizability interpretation á la Kleene in Aczel's constructive set theory [1] extended with (co)inductive definitions [2] (see also [1, 10]).

In this work, extending a previous joint work with M.E. Maietti [7], we show that such results can be improved to a direct correspondence; namely, we show that inductive basic covers (resp. coinductive positivity relations) are equivalent to inductive definitions (resp. coinductive definitions) over the simple Minimalist Foundation. Our work then makes use of the minimality of the Minimalist Foundation to extend the results in [4], and compare the (co)inductive definitions of the Minimalist Foundation with those available in other foundational theories, in particular with W-types and M-types of Martin-Löf's type theory [9] in terms of (co)algebras of polynomial functors. Not surprisingly, this shows that the Minimalist Foundation with inductive and coinductive definitions is exactly what is needed to state and prove all the results of formal topology.

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The variety of Boolean algebras of dimension n and its higher dimensional propositional logic

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SPEAKER: Antonino Salibra.

A research programme developed over the last few years [9, 5, 8, 3, 7, 2, 1] has attempted to single out the features of Boolean algebras that are responsible for them to encompass virtually all the “desirable" properties one can expect from a variety of algebras. The main results obtained so far on this topic revolve around the notion of Boolean algebra of dimension n ($n\mathrm{BA}$) and of the corresponding propositional logic. We provide an overview of these results, with particular attention to recent developments related to the propositional calculus of dimension n.

An n-partition of a set A is a tuple $(A_1, \dots, A_n)$ of pairwise disjoint subsets of A such that $\bigcup_{i=1}^n A_i = A$. The set of n-partitions of A becomes an nBA by endowing it with n constants $\mathsf{e}_1 = (A, \emptyset, \dots, \emptyset), \mathsf{e}_2, \dots, \mathsf{e}_n$ and an $(n+1)$-ary operator q as follows, for all n-partitions $\mathbf{y}^0, \dots, \mathbf{y}^i = (Y_1^i, \dots, Y_n^i), \dots \mathbf{y}^n$:

$$q(\mathbf{y}^0, \mathbf{y}^1, \dots, \mathbf{y}^n) = (\bigcup_{i=1}^{n}(Y_i^0 \cap Y_1^i), \dots, \bigcup_{i=1}^{n}(Y_i^0 \cap Y_n^i)).$$

The n-BAs are the abstract counterpart of the algebra of the n-partitions exactly in the same way as the Boolean algebras are the abstract counterpart of the powerset of a set. By the way, the powerset of A is nothing but the set of 2-partitions of A in disguise. Similarly, the Boolean algebra of universe $\{0, 1\}$ of classical truth values is replaced in the n-ary case by the nBA of universe $\{1, ..., n\}$, representing the generalised truth values.

Varieties of $n\mathrm{BAs}$ happens to share many remarkable properties with the variety of Boolean algebras. In particular, as showed in [7, 2]:

- All the elements of an nBA $\mathbf{A}$ induce a decomposition of $\mathbf{A}$ in n factors.
- all subdirectly irreducible $n\mathrm{BAs}$ of type τ have cardinality n; moreover, any $n\mathrm{BA}$ of type τ is a subdirect product of algebras of cardinality n;
- any pure $n\mathrm{BA}$ (i.e. any nBA having no operation but $q, \mathsf{e}_1, \dots, \mathsf{e}_n$) is a subdirect power of the unique pure $n\mathrm{BA}$ $\mathbf{n}$ of cardinality n.
- for every $n \geq 2$ and type τ, all $n\mathrm{BAs}$ of type τ having cardinality n are primal. Moreover, every variety generated by an n-element primal algebra is a variety of $n\mathrm{BAs}$.

In [2] the theory of n-BAs is put to good use to yield an extension to arbitrary semirings of the technique of Boolean powers. We define the semiring power $\mathbf{A}[\mathbf{R}]$ of an algebra $\mathbf{A}$ by a semiring $\mathbf{R}$, and show that any pure nBA $\mathbf{A}$ is a retract of the semiring power $\mathbf{A}[\mathbf{B_A}]$ of $\mathbf{A}$ by what we call the inner Boolean algebra $\mathbf{B_A}$ of $\mathbf{A}$. Foster's celebrated theorem on primal algebras [4] follows as a corollary from this result.

In [3] the connection between a noncommutative version of Boolean algebras, called skew Boolean algebras [6], and pure nBAs is explored. It is shown that any nBA is obtained by appropriately merging n isomorphic skew Boolean algebras.

In [7] we focus on an application to logic. Just like Boolean algebras are the algebraic counterpart of classical propositional logic CL, for every $n \geq 2$ we define a logic $n\mathrm{CL}$ whose algebraic counterpart are $n\mathrm{BAs}$. We also prove that every tabular logic with a single designated value is a sublogic of some $n\mathrm{CL}$.

In [1] we have started to investigate the proof theory of $n\mathrm{CL}$, by using the sequent calculi $n\mathrm{LK}$. We have shown the soundness and completeness of these calculi. Additionally, as a corollary of a syntactic proof of completeness, we have obtained a result of cut elimination.

A peculiar characteristic of $n\mathrm{LK}$ is that it has a unique connective, q, of arity $n + 1$. The intended meaning of this connective is the following: the truth value of the formula $q(F, G_1, \ldots, G_n)$ is that of G_k, k being the truth value of F. When $n = 2$, this is the usual interpretation of $\mathrm{if\_then\_else}(\mathrm{F}, \mathrm{G}_1, \mathrm{G}_2)$.

Another distinctive feature of this deductive system is that each truth valus i has its own turnstile $\vdash_i$. In the two dimensional case, this gives rise to the turnstiles $\vdash_1$ and $\vdash_2$, the former deriving tautologies, and the latter deriving contradictions. Entailements in the various dimensions are symmetric. In the case $n = 2$, every formula F is equivalent to $q(F, \mathsf{e}_1, \mathsf{e}_2)$, whereas $\neg F$ is equivalent to $q(F, \mathsf{e}_2, \mathsf{e}_1)$. In the general case, $q(F, \mathsf{e}_1, \ldots, \mathsf{e}_n)$ is equivalent to F and there are $\binom{n}{2}$ way of permuting two truth values, yelding $\binom{n}{2}$ different negations.

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Decision problems of group and semigroups as equivalence relations

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SPEAKER: Luca San Mauro.

The study of word problems dates back to the work of Dehn in 1911. Given a recursively presented algebra A, the word problem of A is to decide if two words in the generators of A refer to the same element. Much is known about the complexity of word problems for familiar algebraic structures: e.g., the Novikov-Boone theorem, one of the most spectacular applications of computability to general mathematics, states that the word problem for finitely presented groups is unsolvable. Yet, the computability theoretic tools commonly employed to measure the complexity of word problems (e.g., Turing or m-reducibility) are defined for sets, while it is generally acknowledged that many computational facets of word problems emerge only if one interprets them as equivalence relations.

In this work, we revisit the world of word problems, with a special focus on groups and semigroups, through the lens of the theory of equivalence relations, which has grown immensely in recent decades. To do so, we employ computable reducibility, a natural effectivization of Borel reducibility.

This talk collects joint works with Uri Andrews, Valentino Delle Rose, Meng-Che Ho, and Andrea Sorbi.

Model theory meets set theory

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SPEAKER: Juan M Santiago.

Set theory originated in the 1870s in the works of George Cantor, answering questions both from real analysis and the infinite. **Model theory** appeared in 1915 in the work of Leopold Lowenheim about elementary substructures and elementary extensions. In the modern presentation of mathematical logic, both areas are extremely related. The goal of this presentation is to share recent results relating both areas.

The first topic is a generalization of first order logic: infinitary logics.

Definition 1. Let κ and λ be regular cardinals and let $\mathcal{L}$ be a first order language. The logic $\mathcal{L}_{\kappa\omega}$ has the same terms and atomic formulas as first order logic and generalises the conjunction operation from finitely many to less than κ many. The logic $\mathcal{L}_{\kappa\lambda}$ extends $\mathcal{L}_{\kappa\omega}$ by generalizing the quantification operation from finitely many to less than λ many.

Infinitary logics appeared in the 1950s. Let us mention that weakly compact and strongly compact cardinals arised in this are as the answer to the following question: what infinitary logics admit compactness theorems? The search for infinitary logics with a model theory similar to that of first order logic allowed for a model theoretic study of theories that are relevant to set theory. As the most prominent example I would mention the following result by Barwise: any countable model of ZF can be end-extended to an (ill-founded) model of $ZFC + V = L$.

We pursue this topic with an emphasis on the relation between classical infinitary logics and forcing, or from another point of view, Boolean valued models. The main idea we present is the following: instead of applying infinitary logics as a mean towards understanding classical Tarski like models (that is, with a 0-1 truth value), we consider infinitary logic the object of study itself and look for a semantic providing a nice model theory. This semantic is precisely given by Boolean valued models and allows to separate the logics $\mathrm{L}_{\kappa\lambda}$ from the logics $\mathrm{L}_{\kappa\omega}$ with a very natural condition, the mixing property. In this context two main theorems are proved: interpolation and omitting types [2].

The second direction is obtained from the model theoretic notion of indiscernibles. These were first introduced by Silver as a way to describe and understand the constructible universe L as well as its theory. Very soon after Silver's result on $0^\sharp$, Solovay [1] realized that the technique could be pursued not only in the model L, but also in $\mathrm{L}(\mathbb{R})$. The main result in this direction, which we will argue how to generalize, is the following: Let A be a transitive set and T the theory of $\mathrm{L}(A)$ in language $\mathcal{L}_A$. Then T has a prime model.

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Universality proprieties of graph homomorphism: one construction to prove them all

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SPEAKER: Salvatore Scampeerti.

An undirected graph G is a pair (V, E), where V is the set of vertices and E is an irreflexive symmetric binary relation on V. We refer to E as the set of edges of G. Given two undirected graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, a graph homomorphism is a function $h : V_1 \to V_2$ such that if $x \ E_1 \ y$ then $h(x) \ E_2 \ h(y)$. The graph homomorphism relation $\preccurlyeq$ is an extensively studied quasi-order. It finds application in several fields due to the possibility of encoding objects and their relationships as graphs and homomorphisms between them. Independently of the field one is working in, a recurring theme is that the graph homomorphism quasi-order is very complicated, or, in other words, it can encode "a lot"; in such cases, one usually says that graph homomorphism is universal. However, the proofs of those universality results are often ad hoc constructions which depend on the framework at hand and cannot be used in other ones. The goal is to unify these different perspectives, designing a standard operation on graphs which allows us to reprove at once many existing universality results, and solve some natural open problems. Despite its simplicity, our technique unexpectedly leads to applications in quite diverse areas of mathematics, such as category theory, structural combinatorics, classical descriptive set theory, model theory, generalized descriptive set theory, and theoretical computer science. This is a joint work with Luca Motto Ros.

Weil 2-rigs

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SPEAKER: Luca Spada.

A *rig* is an algebra $\langle R, +, \cdot, 0, 1\rangle$ where $\langle R, +, \cdot\rangle$ is a commutative semiring in which 0 and 1 are the neutral elements for $+$ and $\cdot$, and multiplication by 0 annihilates R, i.e. $0 \cdot x = 0$ for any $x \in R$. We call 2-*rig* any additively idempotent rig.

It has been maintained (see [2, 4]) that the category of 2-rigs has many things in common with the category of k-algebras for an algebraically closed field k. From this perspective, the theory of 2-rigs has a very geometric flavor. The reason for this claim is that both categories are *co-extensive*, i.e. dual to an extensive category. A category C with finite coproducts is *extensive* [1] if the canonical functor $\mathsf{C}/X \times \mathsf{C}/Y \to \mathsf{C}/(X + Y)$ is an equivalence for every pair of objects X, Y in C. Extensivity describes a basic property of coproducts in categories 'of spaces', with many interesting consequences. For instance, while the category of topological spaces and continuous functions between them is extensive, the category of groups is not.

The reason for considering the concept of Weil 2-rigs, which we are about to introduce, is in accordance with the above proposal. Let C be a category with a terminal object 1. If X is an object of C, a *point* of X is an arrow $1 \to X$. It is well known that there may exist several non-isomorphic objects with only one point. As proved in [3], the finitely generated local complex algebras that have a unique map into the (initial algebra) $\mathbb{C}$ are the so-called Weil algebras; intuitively, these are function algebras of spaces with a single point. In this work, we are concerned with similar objects in the category of 2-rigs.

Definition 1. A 2-rig R is called *Weil* if there exists a unique homomorphism from R into the initial 2-rig **2**.

We present here two characterizations of Weil 2-rigs: the first is in terms of its prime ideals, the second is an explicit description in geometric logic. Recall that in any 2-rig one can define a partial order by setting $x \leq y$ iff $x + y = y$.

Theorem 2. *Let R be any 2-rig. Then the following are equivalent.*

1. *R is a Weil 2-rig.*
2. *R has a unique prime ideal downward-closed with regard to $\leq$.*
3. *For all $x \in R$ there exits $n \in \mathbb{N}$ such that $x^n \leq 0$ or $1 \leq rx$ for some $r \in R$.*

The characterization of Weil 2-rigs in terms of prime ideals also allows us to prove the following result.

Theorem 3. *Weil 2-rigs form a coreflective subcategory of 2-rigs.*

Following the geometric guidance, the next definitions are again natural. An arrow $f\colon X \to Y$ in C is called *constant* if it factors through the terminal object. More generally, an arrow $f\colon X \to Y$ is called a *pseudo-constant* if it coequalizes all the points of X; in other words: for every pair of points $a, b\colon 1 \to X$, one has $f(a) = f(b)$.

1 [transform canvas=yshift=1ex, r, "b"] [transform canvas=yshift=-1ex, r, "a"] X [r, "f"] Y

Of course, every constant arrow is pseudo-constant. Dualizing, we obtain:

Definition 4. A homomorphism of 2-rigs $f\colon A \to B$ is called *pseudo-stant* if for each pair $g, h\colon B \to 2$ one has $g \circ f = h \circ f$.

In Set pseudo-constants maps are constant, but this is not true, in general, in categories of spaces. Using Theorem 3 we prove that in the opposite category of 2-rigs the image of any pseudo-constant map has exactly one point.

Corollary 5. *Any pseudo-stant arrow in* 2Rig *factors through a Weil 2-rig.*

Again, guided by geometric intuition, one might conjecture that, for a 2-rig A, the family of all maps from Weil 2-rigs into A is jointly monic. Unfortunately, this is false, as a finite example shows. Nevertheless, finitely generated free 2-rigs have this property, as recorded in the following result.

Theorem 6. *Every finitely generated free 2-rig is a subdirect product of finite Weil 2-rigs.*

From this it can be easily deduced the following corollary.

Corollary 7. *The variety of 2-rigs is generated by finite Weil 2-rigs.*

We conclude by noticing that all the aforementioned results also apply to the subvariety of 2-rigs formed by *integral rigs*, i.e, rigs that satisfy the *integrality* law $x + 1 = 1$.

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Higher dimensional semantics of propositional theories of dependent types

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SPEAKER: Matteo Spadetto.

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In recent years, there has been a growing interest in various weakenings for theories of dependent types, particularly those weakenings with respect to the strength of the computation rules of the type constructors. When a dependent type theory has a type constructor that relies on a propositional equality instead of a judgemental equality, as is the case in Martin-Löf Type Theory, one says that the type constructor is in *propositional* form. Thus, a dependent type theory will have e.g. *propositional* identity types if it is endowed with a type constructor satisfying the usual rules of intensional identity types, except for the judgemental equality of its computation rule: whenever we are given judgements $x, y : A;\ p : x = y \vdash C(x, y, p) : \textsc{Type}$ and $x : A \vdash q(x) : C(x, x, r(x))$, in place of asking that the judgement $x : A \vdash J(x, x, r(x), q) \equiv q(x)$ holds -here J denotes the identity type eliminator-, we only ask that a judgement:

$$x : A \vdash H(x, q) : J(x, x, r(x), q) = q(x)$$

holds; see [2, 3] for more details.

Coquand and Danielsson [2] were the first to consider propositional identity types. This type constructor has since been extensively studied by van den Berg [3], Bocquet [4], Spadetto [6] and others. One might consider the same form of weakening for the computation rule of dependent sum types and dependent product types: these type constructors satisfying a propositional computation rule will be called *propositional dependent sum types* and *propositional dependent product types* respectively.

In this talk, we discuss a dependent type theory that includes propositional identity types, propositional dependent sum types, and propositional dependent product types, along with an arbitrary family of atomic types. We will refer to such a theory as a *propositional type theory*. The aim of this talk is to show how such a calculus admits a natural notion of semantics - generalising the one presented in [5] - via 2-dimensional categorical structures called *display map 2-categories*, where propositional type constructors are encoded as 2-dimensional categorical properties. We compare the class of models according to this semantics with the class of those derived from the usual notion of semantics for theories of dependent types, which is typically phrased via *display map categories* or *categories with attributes* (see [1]).

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Logic, Logicality, and Arithmetical Determinacy

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SPEAKER: Sebastian G. W. Speitel.

According to Tarski's celebrated *model-theoretic definition of logical consequence* a sentence φ follows logically from a class of sentences Γ if and only if every model of the sentences in Γ is also a model of the sentence φ [8]. While explicating the notion of logical consequence for a wide range of languages the definition relies on a prior division of the expressions of the language of φ and Γ into logical and extra-logical: the logical constants of a language directly influence the range of admissible models to take into account when assessing a claim of consequence and thereby which arguments are logically valid.

Since logical consequence is assumed to be a *formal* relation, i.e., one that is not influenced by extraneous 'empirical' information, the choice of which expressions to count as logical is not arbitrary: it must be such as to ensure that inferences licensed on the basis of that choice respect the constraint of formality. Determining the right boundary between logical and non-logical constants is thus of crucial importance to the Tarskian project of providing an adequate model-theoretic explication of the notion of logical consequence. If too many, or inappropriate, expressions are classified as logical several 'material', non-formal transitions will, wrongly, qualify as logical consequences; if too few expressions are taken to be logical the resulting relation of logical consequence will be impoverished and not yield an adequate account for a given language. This is the *demarcation problem of the logical constants*.

Although it is easy enough to provide a satisfactory division of the required kind for common logical languages, such classifications often proceed by means of uninformative enumerations. This unprincipled and case-by-case determination contrasts sharply with the generality of the definition of logical consequence which applies to all languages of a particular type indiscriminately. Tarski himself regarded the project of providing a mathematically rigorous explication of the notion of logical consequence as unfinished until this crucial issue could be addressed. What is needed to put the model-theoretic definition of logical consequence on a firm philosophical foundation and shield it against skeptical attacks is, therefore, a *criterion of logicality*, a set of mathematically precise and philosophically informative principles which delineate, for the kinds of languages covered by Tarski's definition, the appropriate class of logical constants.

In the tradition of devising criteria to solve the demarcation problem of the logical constants *invariance-based approaches* hold a prominent place ([9], [7], [6], [1]). These criteria ground the formality of logical inferences in properties of the model-theoretic denotations of purported logical constants. Despite oftentimes regarded as a *necessary* component of delineating the logical expressions of a language, purely invariance-based criteria appear to face issues they are, by their very design, unable to overcome. Criteria of this sort for the most part address only the *semantic question* of what constitutes a logical meaning while neglecting the attendant *meta-semantic question* of how logical constants come to possess such meanings.

The goal of this talk is to present, motivate and defend a novel criterion of logicality which supplements invariance-based constraints with inferentialist requirements on the determination of meaning and to explore its scope and some of its consequences. Central to the proposed criterion is a combination of insights from two traditions in the philosophy of logic and language which, together, address both the semantic and meta-semantic question: from the model-theoretic tradition it adopts the idea that the formality of logical consequence is grounded in properties of the denotation of logical expressions, best captured by an invariance constraint. From the inferentialist tradition it takes up the insight that the meaning of a logical expression should be recoverable from its inferential behaviour, that its meaning should be *uniquely determined by its inferential role*. This is operationalized by means of a *categoricity-requirement*. The resulting criterion demands that for an expression to be logical the inferential and model-theoretic aspects of its meaning must cohere in such a way that its inferential behaviour (codified by a consequence relation) uniquely determines one among its consistent, formal – i.e., (isomorphism-)invariant, – model-theoretic values (see [2]).

After motivating a combined criterion of this kind I assess how it fares with respect to expressions from the category of generalized quantifiers. Based on a recent result by Bonnay & Westerståhl [3] one can show that the criterion indeed covers the standard classical quantifiers of FOL (a rather minimal standard of adequacy). I then proceed to investigate its scope and limits by means of two types of examples (restricting consideration to type $\langle 1 \rangle$ quantifiers):

(1) the quantifier $\mathcal{Q}_0$, 'there are infinitely many' and, relatedly, also the quantifier 'there are finitely many' qualify as logical according to the proposed criterion.

(2) the quantifier $\mathcal{Q}_1$, 'there are uncountably many' and various of its relatives of the form $\mathcal{Q}_\alpha$ ($\alpha > 1$) do not qualify as logical under the proposed criterion.

If there is time, I will consider further examples of quantifiers that (do not) qualify as logical according to the criterion and present some results regarding their unique determination (relating to preservation of this property under, e.g., boolean combinations, EC_Δ-definability, etc.).

In the last part of the talk I want to pursue a particular consequence of the criterion and explore the resulting conception of logic a bit further: it is well-known that the natural number structure can be categorically characterized in the language of FOL extended by the quantifier 'there are (in)finitely many'. I will here consider the question in how far the determinate reference to the natural number structure achieved on the basis of uniquely determinable notions provides an effective reply to the 'model-theoretic skeptic' in the philosophy of mathematics (see, e.g., [4]) who doubts our ability to achieve, in a naturalistically acceptable way, such reference.

This talk incorporates joint work with Denis Bonnay.

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Subdirectly irreducible idempotent and integral rigs

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SPEAKER: Gavin St. John

A special class of semirings, often referred to as *rigs*, are those semirings in which both operations are commutative and where the additive identity is multiplicatively absorbing.* More formally, a rig is an algebra $\mathbf{R} = \langle R, +, \cdot, 0, 1\rangle$ satisfying $x \cdot (y + z) \approx (x \cdot y) + (x \cdot z)$ and $0 \cdot x \approx 0$, where both $\langle R, +, 0\rangle$ and $\langle R, \cdot, 1\rangle$ are commutative monoids. Rigs are rather general structures, encompassing e.g., commutative rings (by taking the appropriate fragment) and even bounded distributive lattices, and are therefore pervasive many disciplines of mathematics, and even computer science, in one form or another.

Of special interest are those (sub-) varieties of additively-idempotent rigs, called *2-rigs*, and integral rigs (i.e., satisfying $1 + x \approx x$), called *irigs*. In fact, the former class subsumes the latter. Since addition is idempotent in these structures, the additive fragment is actually a semilattice with an induced a partial order in which 0 is the least element, and

*The name coming from the pun"rng" ("ring" without **i**dentity), where a "rig" is a "ring" without **n**egatives in the case of irigs, also has the constant 1 as the greatest element. In particular, bounded distributive lattices are exactly those irigs in which multiplication is (also) idempotent.

While the names "2-rigs" and "irigs" originate from algebraic-geometric considerations, they also constitute the $\{\vee, \cdot, 1, 0\}$-fragments of (commutative) residuated structures, namely $\mathsf{FL}_{\mathsf{eo}}$ and $\mathsf{FL}_{\mathsf{ew}}$-algebras, respectively. Such classes of algebras (and their subvarieties) form the equivalent algebraic semantics, in the sense of Blok and Pigozzi, for some of the most interesting substructural logics. It is therefore of general interest studying this fragment, as it naturally corresponds to structural properties of those logics. For instance, the (algebraic) identities in this signature correspond to structural (inference) rules in the logics.

In this work we study some basic universal-algebraic properties of rigs, with special emphasis placed on the idempotent and integral cases. In particular, for which we provide the following characterization of subdirectly irreducible irigs:

Proposition 1. *An irig $\mathbf{R}$ is subdirectly irreducible if and only if the following hold:*

1. *There exists an element $\epsilon \in R$, called* the second element*, such that $\epsilon \leq r$ for any $r \neq 0$;*
2. *Each distinct pair of elements $a, b \in R$ can be* separated*, meaning $a \cdot r \neq b \cdot r$ with $0 \in \{a \cdot r, b \cdot r\}$ for some $r \in R$.*

Given the above characterization, it is not difficult to see that the monolith of a subdirectly irreducible irig coincides with the set of pairs $\Delta \cup \{(0, \epsilon), (\epsilon, 0)\}$, i.e., the only elements that are "collapsed" are zero and the second element. More generally, we call a pair of distinct elements a, b in an algebra a *clique* if the congruence generated by the pair collapses only the pair a, b. Guided by the arguments used to establish Proposition 1, a more general theorem for rigs arises. In the context of 2-rigs, it can be stated more simply as follows:

Theorem 2. *Let $\mathbf{R}$ be a 2-rig. Then the following are equivalent.*

1. *$\mathbf{R}$ is subdirectly irreducible whose monolith arises from a clique C.*
2. *There exists distinct $a, b \in R$, with $C := \{a, b\}$, and the following hold:*
   - *(a) $a \leq b$*
   - *(b) $x \not\leq a$ implies $a + x = b + x$;*
   - *(c) $ax \neq a$ implies $ax = bx$; and*
   - *(d) For any distinct pair $c, d \in R$ there is $r \in R$ such that either $cr \leq a < dr$ or $dr \leq a < cr$.*

It should be noted, however, that the above is only a *sufficient* characterization for subdirectly irreducible 2-rigs, as there are examples whose monolith is more complex. Nevertheless, the subdirectly irreducible 2-rigs of this form are enough to generate the variety:

Theorem 3. *The variety of 2-rigs is generated by those of (finite) subdirectly irreducible algebras arising from cliques.*

Moreover, the class mentioned in the above theorem can be specialized even further to those algebras that additionally admit a certain "geometric" flavor; the so-called *Weil* 2-rigs.

Elementarity and cardinal correctness

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SPEAKER: Sebastiano Thei.

Sebastiano Thei was partially supported by the Italian PRIN 2017 Grant "Mathematical Logic: models, sets, computability," by the Italian PRIN 2022 Grant "Models, sets and classifications" and by the European Union - Next Generation EU.

Large cardinal axioms are typically formulated in terms of elementary embeddings from the universe V into some transitive subclass M. By demanding a stronger and stronger degree of resemblance between V and M, one obtains stronger and stronger principles of infinity. In the early 1970s, Kunen [3] refuted the natural extreme of this trend by proving that there is no nontrivial elementary embedding from the universe to itself. Straining the limits of consistency, Caicedo [2] proposed a new way to extend the large cardinal hierarchy in ZFC and to obtain axioms just at the edge of Kunen inconsistency:

Definition 1. An elementary embedding $j : M \to N$ between two transitive models of *ZFC* is *cardinal preserving* if M and N correctly compute the class of cardinals.

Even though the expectation is that ZFC rules out cardinal preserving elementary embeddings, their existence is still unclear. Taking the first step towards this line of research, we consider the case where either M or N is V. More specifically, we show that cardinal preserving elementary embeddings $j : M \to V$ are inconsistent with ZFC, answering a question of Caicedo. The proof involves singular cardinal combinatorics and relies on some results concerning square principles (Magidor-Sinapova [4]), good scales, Jónsson cardinals (Shelah [5]), ω_1-strongly compact cardinals (Bagaria-Magidor [1]) and basic facts from Shelah's pcf theory. We also discuss the case $M = V$ and look at the influence that a cardinal preserving elementary embedding $j : V \to N$ has on the universe. It is still open whether ZFC alone can refute such a j. This is joint work with Gabriel Goldberg.

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Universal properties in computability: a categorical perspective

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SPEAKER: Davide Trotta.

One of the most important notions of categorical logic which enables the study of logic from a pure algebraic perspective is that of a *hyperdoctrine*, introduced in a series of seminal papers by F.W. Lawvere [5, 4] in the 1969.

Over the years, several authors have developed, applied and generalized this notion in many areas of mathematics. Among these, a field that has particularly benefited from this categorical approach, resulting in a substantial and extensive body of literature (as expounded, for example, in van Oosten's book [14]), is that of realizability interpretation of intuitionistic logical systems.

Despite the long tradition of studying realizability and its variants from a categorical perspective, there is still a lack of a systematic and in-depth analysis of computability-like notions. This analysis would begin with the basic notion of recursively enumerable predicates (r.e.) and extend to include Turing [9, 10], Medvedev [11, 2], Muchnik [2], and Weihrauch reducibilities [1].

The main purpose of this talk is to explain how we can broaden the application of categorical tools to computability. This is achieved by introducing a specific doctrine for r.e. (playing a crucial role in the construction of Joyal's arithmetic universes [7]) and for each previously mentioned notion of reducibility, and investigating their abstract and common universal properties.

Indeed, by citing results from [8] and [13], we demonstrate that all the aforementioned notions of reducibility emerge as instances of either of the following two categorical constructions: the universal completion and the existential completion [12, 6].

These results align with Hofstra's analysis of realizability-like doctrines [3], aiming to show that all the doctrines used to build realizability-like toposes are instances of a free construction adding "generalized existential quantifiers".

The long-term goal of our analysis is to establish a common language for the categorical logic community and the computability theorists. Traditionally, these two fields have used very distinct languages and notations. Therefore, we believe that by providing categorical explanations of fundamental concepts in computability, framed within the context of doctrines, we can encourage and strengthen collaborations and interactions between these two communities.

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Equational anti-unification, algebraically

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SPEAKER: Sara Ugolini.

A synctactic *anti-unification* problem consists in finding, for a fixed finite set of terms, another term s that is a common *anti-unifier* (or *generalizer*) for all of them; more formally, given a finite set of terms $t_1, \ldots, t_k$ over some algebraic language, finding a *anti-unifier* for them means finding a term s on the same language and substitutions $\sigma_1, \ldots, \sigma_k$ such that for all $i = 1, \ldots, k$, $\sigma_i(s) = t_i$. The problem of finding an anti-unifier of a set of terms has first been formalized around 1970 by Popplestone [5], Plotkin [4], and Reynolds [6], in order to study inductive proofs. In more recent years, anti-unification has been applied to several domains, such as inductive logic programming, case-based reasoning, conceptual blending.

In this contribution we are interested in anti-unification *up to some equational theory* $\mathcal{E}$ (also called *equational anti-unification*); i.e., given an anti-unification problem $\{t_1, \ldots, t_k\}$, a solution is given by a term s for which there exist $\sigma_1, \ldots, \sigma_k$ such that for all $i = 1, \ldots, k$, $\mathcal{E} \models \sigma_i(s) = t_i$.

Given the fact that anti-unification problems always have a solution (mapping a fresh variable to each term), it becomes relevant to know whether there is a *least general solution*: a solution that can be obtained by all other possible solutions by further substitution. More generally, one can define an order between solutions: given two solutions s and s', s is *less general* than s' if and only if there exists a substitution σ such that $\sigma(s') = s$; this gives a preorder, and if one considers the associated equivalence relation, the corresponding equivalence classes (i.e., the anti-unifiers that are "equally general") form a poset. Then one can define a notion of *anti-unification type*, which intuitively gives the cardinality of the set of least general solutions. Given an equational theory $\mathcal{E}$, its anti-unification type is the worst possible type occurring among all its anti-unification problems.

The study of equational anti-unification and its type, depending on the equational theory, have recently received attention (see a recent survey [2]); while an interest for theoretical foundations seems to be growing, no general approach has yet been developed within the universal algebraic framework, which arguably is the most natural framework to deal with models of equational theories. We aim at filling this void.

We note that in contrast, the study of the "dual" of anti-unification, namely *unification problems* and their type, have been successfully studied in the general algebraic framework via the approach developed by Ghilardi [3]. In a unification problem, given by a finite set of terms $t_1, \ldots, t_k$, one looks for a *unifier*, i.e. a substitution σ for which $\sigma(t_1) = \ldots = \sigma(t_k)$ for all $i = 1, \ldots, k$. Ghilardi translates the investigation of unification problems (up to an equational theory) to the algebraic setting via the use of finitely presented and projective algebras in the variety determined by the equational theory.

We shall see that, in analogy with Ghilardi's approach, also in the case of anti-unification problems one makes use of projective algebras in order to formalize solutions in the algebraic setting. However, we shall also note that the two problems of unification and anti-unification have quite a different nature; this difference is apparent when one studies the poset of solutions. Indeed, while in the unification case the order is really among the substitutions that yield the solution, in the case of anti-unification the order is within the *terms*. This difference yields the necessity, in the algebraic setting, to consider *pointed* algebras. In detail, given a variety V, we formalize its anti-unification problems using its *pointed companion* V_p, i.e. the class of *pointed algebras* $(a, \mathbf{A})$ given any $\mathbf{A} \in \mathsf{V}$ and $a \in A$. V_p constitutes a category whose morphisms are *pointed homomorphisms*, that is to say homomorphisms $h : (a, \mathbf{A}) \to (b, \mathbf{B})$ such that $h(a) = b$.

Definition 1. We say that an *algebraic anti-unification problem* for a variety V is a pointed algebra $((t_1, \ldots, t_k), \mathbf{F}_{\mathsf{V}}(X)^k)$ in the pointed companion V_p of V, where X is the finite set of variables appearing in $t_1, \ldots, t_k$, and $\mathbf{F}_{\mathsf{V}}(X)$ is the free algebra in V over the set of generators X. An *algebraic solution* for such a problem is a pointed algebra $(p, \mathbf{P}) \in \mathsf{V}_p$ such that $\mathbf{P}$ is finitely generated and projective in V and there exists a pointed homomorphism $g : (p, \mathbf{P}) \to ((t_1, \ldots, t_k), \mathbf{F}_{\mathsf{V}}(X)^k)$.

We define a generality order on solutions as follows. A solution $(p, \mathbf{P})$ is said to be *less general* than a solution $(p', \mathbf{P}')$ iff there exists a pointed homomorphism $h : (p', \mathbf{P}') \to (p, \mathbf{P})$.

Theorem 2. *A symbolic anti-unification problem $t_1, \ldots, t_k \in \mathbf{F}_{\mathsf{V}}(X)$ has a solution if and only if the algebraic problem $(t_1, \ldots, t_k, \mathbf{F}_{\mathsf{V}}(X)^k)$ does; moreover their posets of solutions are isomorphic.*

A key idea of this approach is to see the terms $t_1, \ldots, t_k$ as a *single element* of an algebra (precisely of the k-th power of the appropriate free algebra). Building on this intuition, we show that $\mathbf{F}_{\mathsf{V}}(1)$, the 1 generated free algebra in the variety under consideration, plays a special role. In particular we show that, under certain assumptions, the study of the anti-unification type can be reduced to the study of the poset of congruences of $\mathbf{F}_{\mathsf{V}}(1)$ that correspond to a projective quotient. A relevant property that we require in order to obtain such a result is for 1-generated subalgebras of free algebras to be projective in the variety, a condition that is satisfied by many relevant varieties related to logic (e.g., Boolean algebras, Heyting algebras, Gödel algebras, Kleene algebras, MV-algebras).

Note that the algebraic approach we presented allows one to study anti-unification problems for logics that are algebraizable in the sense of Blok and Pigozzi [1]. Consider a logic L that is Blok-Pigozzi algebraizable and let Q be the quasivariety that is its equivalent algebraic semantics. There is a natural way to associate to L an equational theory:

consider $\varphi \approx \psi$ iff φ and ψ are logically equivalent, i.e., φ and ψ have the same interpretation in every model of the logic. By algebraizability this is exactly the equational theory of the variety generated by Q. The situation simplifies if L is *strongly* algebraizable, i.e., Q is a variety. Hence, algebraizable logics inherit the notion of anti-unification problem and anti-unification type in the sense that they are the ones of their corresponding variety of algebras. We use our methods to study the anti-unification problems and their type for: classical logic, Gödel-Dummet logic, Kleene's 3-valued logic, 3-valued and infinite-valued Łukasiewicz logics.

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Quotients in the Weihrauch degrees

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SPEAKER: Manlio Valenti.

In [1], the authors introduced the parallel quotient operator on computational problems, mapping two partial multi-valued functions f, g to

$$f/g := \sup_{\equiv_{\mathrm{W}}}\{h : g \times h \leq_{\mathrm{W}} f\}.$$

They used this notion to study the interplay between Ramsey-theoretical problems as RT^2_2 or SRT^2_2 and other well-known computational problems like LPO and NON. However, many of the properties of this operator are left unexplored; most notably, they raised the question of whether the quotient operator is total.

We answer this question affirmatively, and provide a canonical representative for such degree. We observe that $0 <_W f/g$ iff $g \leq_W^* f$, which in turn implies that the continuous Weihrauch degrees are first-order definable in $(\mathcal{W}, \leq_W, \times)$. We also observe that f/g is pointed iff $g \leq_W f$ and that $f^*/g \equiv_W f^* \times d_A$ where $A \subseteq \mathbb{N}^\mathbb{N}$ is the (possibly empty) set of oracles witnessing $g \leq_W^* f^*$.

Intuitively, f/f calibrates how close is f to being closed under parallel product. Indeed, $f/f \equiv_W f$ iff $f \times f \leq_W f$. On the other hand, there are many explicit examples of problems for which $f/f \equiv_W \mathrm{id}$, like C_k, LPO^k, RT^1_k. Unsurprisingly, there are degrees exhibiting intermediate behaviors. In particular, the quotient operator can be used to neatly describe some recent results [3] on the uniform computational strength of the problem DS of finding descending sequences through ill-founded linear orders (originally introduced in [2]). In particular, we show that

Theorem 1.

1. $\mathsf{DS}/\widehat{\mathsf{ACC}_\mathbb{N}} \leq_W \lim$
2. $\mathbf{\Pi}^1_1\text{-}\mathsf{Bound} <_W \mathsf{DS}/\mathbf{\Pi}^1_1\text{-}\mathsf{Bound} \leq_W \mathsf{DS}/\mathsf{RT}^1_2 <_W \mathsf{DS}$
3. $\mathsf{DS}/\mathsf{C}_\mathbb{N} \equiv_W \mathsf{DS}$

This is joint work with Jun Le Goh and Arno Pauly.

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Reverse mathematics and dimension of posets

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SPEAKER: Andrea Volpi.

There are many different parameters used in order theory to describe a poset. One of them is the dimension, which is the least cardinality of a set of linearizations whose intersection is the starting poset. With such definition, linearly ordered sets have dimension 1 while antichains have dimension 2. I will introduce order theory and I will state bounding theorems about the dimension of posets with many examples. Then I will talk about the strength of these results from the point of view of Reverse Mathematics. This is joint work with Alberto Marcone and Marta Fiori Carones.

Intuitionistic Metric Temporal Logic

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SPEAKER: Margherita Zorzi.

Baratella and Masini [1, 2, 3] previously explored, from a proof-theoretic perspective, a propositional and a predicate sequent calculus featuring an ω-type schema of inference, which naturally interpret the propositional and predicate until-free aspects of Linear Time Logic (**LTL**). The present abstract aims to undertake similar investigations within an intuitionistic framework, obtaining an infinitary Intuitionistic Metric Temporal Logic (**IMTL**). In this preliminary investication we will focus on the discrete case.

The idea is to adapt the labelled sequent calculus in [2]. The intended reading of a labeled formula $t\colon A$ is "A holds at time t".

The relational part is based on Heyting Arithmetic, then the relational language contains $\overline{0}, \overline{s}, \overline{+}.\overline{\cdot}, =, \leq$ as primitive symbols. In particular, f will take the place of the successor function, which we write as $\lambda x.x + 1$, and thus $f^m(t) = t + m$. For the sake of simplicity, we consider the terms as equivalence classes modulo associativity and commutativity.

As in [2] the (propositional) temporal language include the modal operators $\Box_{[m,n]}$, $\Box_{(m,\omega)}$, $\Diamond_{[m,n]}$, $\Diamond_{(m,\omega)}$

We here focus here on the temporal operator *next* $\bigcirc$, definable as $\bigcirc = \Box_{[1,1]} = \Diamond_{[1,1]}$ via the rules

$$\frac{\Gamma, t+1\colon \alpha \vdash \xi}{\Gamma, t\colon \bigcirc\alpha \vdash \xi}\,\mathrm{L}\bigcirc \qquad\qquad \frac{\Gamma \vdash t+1\colon \alpha}{\Gamma \vdash t\colon \bigcirc\alpha}\,\mathrm{R}\bigcirc.$$

Of course, $t\colon \bigcirc A$ is to be read as "A holds at time $t+1$." We also have a rule of *temporal induction*

$$\dfrac{\Gamma, t+x\colon \alpha \vdash t+(x+1)\colon \alpha}{\Gamma, t\colon \alpha \vdash t+u\colon \alpha}\,\text{t.i.}$$

with the condition that x is not in t, u and does not occur freely in Γ.

The main question we are asking is whether there exists a Hilbert-style system for **IMTL**. To this end, we derived all the axioms of the intuitionistic $S4$ (see [4]).

As a title of example, we sjow the proof of the temporal *induction* principle expressed in the following form:

$$\vdash t\colon A \wedge \Box(A \to \bigcirc A) \to \Box A.$$

The proof proceeds as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{t \le t+x, t+x\colon A, (t+x)+1\colon A \vdash t+(x+1)\colon A}}{t \le t+x, t+x\colon A, t+x\colon \bigcirc A \vdash t+(x+1)\colon A}\,\mathrm{L}\bigcirc \qquad \dfrac{}{t \le t+x, t+x\colon A \vdash t+x\colon A}}{t \le t+x, t+x\colon A, t+x\colon A \to \bigcirc A \vdash t+(x+1)\colon A}\,\mathrm{L}\to}{t \le t+x, t\colon A, t+x\colon A \to \bigcirc A \vdash t+z\colon A}\,\text{t.i.}}{t \le t+x, t\colon A, t\colon \Box(A \to \bigcirc A) \vdash t+z\colon A}\,\mathrm{L}\Box}{t\colon A, t\colon \Box(A \to \bigcirc A) \vdash t+z\colon A}\,\mathrm{Mono}_+}{t\colon A \wedge \Box(A \to \bigcirc A) \vdash t+z\colon A}\,\mathrm{L}\wedge}{t\colon A \wedge \Box(A \to \bigcirc A) \vdash t\colon \Box A}\,\mathrm{R}\Box}{\vdash t\colon A \wedge \Box(A \to \bigcirc A) \to \Box A}\,\mathrm{R}\to$$

Notice that we are allowed to use $\vdash t \le t+x$ since we are assuming the temporal part is based on Heyting arithmetic. To understand if a weaker arithmetic could be enough is a question we aim to answer to.

Another interesting question we plain to answer to is to understand what principles are not admissible. For example, we claim that the dual of the temporal induction is not derivable.

Concerning the semantics, in our opinion what deeply characterizes a purely constructive approach is the BHK interpretation. Consider the intuitionistic model for the discrete linear temporal logic $\mathcal{M}_{\mathsf{V}} = \langle Nat, s, 0, \mathsf{V}\rangle$, where we assume to have define the "$\le$" relation in terms of primitive recursive subtraction and to have an effective enumeration of all propositional symbols. The intended meaning of the total recursive evaluation function $\mathsf{V} : Nat \times Nat \to \{0,1\}$ is the following: $\mathsf{V}(n.m) = 1$ means that we have a proof that the propositional symbol p_n is true at m. The BHK interpretation is standard for propositional connectives. Let's see some examples for temporal connectives. A proof of a formula $t : \bigcirc A$ is a a proof of $t+1 : A$; a proof of $t : \Box_{[m,I)} A$, with $I \le \omega$ is a costruction f that transforms a proof π of $t \le m+p < I$ in a proof $f(\pi)$ of $p : A$; a proof of $t : \Diamond_{[m,I)} A$, with $I \le \omega$ is a triple $\langle f, g, p\rangle$ such that g is a proof of π of $t \le m+p < I$ and f is a proof of $t : A$.

On this basis, we aim to interpret by purely constructive arguments, all the LTL axioms. For example, form a preliminary inspection, we noticed that the proof of the inductive principle is exactly what Heyting shows in [5]. A mandatory step will be to prove a soundness theorem w.r.t. the given semantics.

From these preliminary considerations, it has emerged that studying a purely constructive approach in the temporal logic setting seems to be a challenging and interesting task that we aim to develop.

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